Norm Inequalities for Composition Operators on Hardy and Weighted Bergman Spaces

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Abstract. Any analytic self-map of the open unit disk induces a bounded composition operator on the Hardy space H^2 and on the standard weighted Bergman spaces A^2_α. For a particular self-map, it is reasonable to wonder whether there is any meaningful relationship between the norms of the corresponding operators acting on each of these spaces. In this paper, we demonstrate an inequality which, at least to a certain degree, provides an answer to this question.

1. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let φ be an analytic self-map of $\mathbb{D}$. If $\mathcal{H}$ is a Hilbert space of analytic functions on $\mathbb{D}$, the *composition operator* C_φ on $\mathcal{H}$ is defined by the rule $C_\varphi(f) = f \circ \varphi$. While there are some Hilbert spaces (the Dirichlet space, for example) on which there are unbounded composition operators, every analytic $\varphi : \mathbb{D} \to \mathbb{D}$ induces a bounded operator on all of the spaces we will be considering in this paper. Our main goal is to develop a better sense of the relationship between the operator norms of C_φ acting on different spaces.

The Hilbert spaces of primary interest to us will be the Hardy space H^2 and the weighted Bergman spaces A^2_α. The *Hardy space* consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|^2_{H^2} = \sup_{0<r<1} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^2 \frac{d\theta}{2\pi} < \infty.$$

This space is a Hilbert space, with inner product

$$\langle f, g\rangle_{H^2} = \lim_{r\uparrow 1} \int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)} \frac{d\theta}{2\pi}.$$

The Hardy space can be described as a *reproducing kernel Hilbert space*, since for every point λ in $\mathbb{D}$ there is a unique function K_λ in H^2 (known as a *reproducing kernel function*) such that $\langle f, K_\lambda\rangle_{H^2} = f(\lambda)$ for all f in H^2. In the case of the Hardy space, it is not difficult to see that $K_\lambda(z) = 1/(1-\overline{\lambda}z)$ (see Corollary 2.11 in [8]).

For $\alpha > -1$, the *weighted Bergman space* A^2_α consists of all analytic f on $\mathbb{D}$ such that

$$\|f\|^2_{A^2_\alpha} = \int_{\mathbb{D}} |f(z)|^2(\alpha+1)(1-|z|^2)^\alpha \, dA(z) < \infty,$$

where dA signifies normalized area measure on $\mathbb{D}$. The case where $\alpha = 0$ is known as the (unweighted) Bergman space, and is often denoted simply A^2. For any α, we write $\langle\cdot,\cdot\rangle_{A^2_\alpha}$ to denote the obvious inner product on A^2_α. These spaces are all reproducing kernel Hilbert spaces, with kernel functions $K^\alpha_\lambda(z) = 1/(1-\overline{\lambda}z)^{\alpha+2}$ (see Corollary 2.12 in [8] and Proposition 1.4 in [11]).

There is an obvious likeness between the reproducing kernels for H^2 and the analogous functions for A^2_α. For the sake of efficiency, it will often behoove us to write A^2_{-1} to denote the Hardy space H^2, with $K^{-1}_\lambda = K_\lambda$ and $\langle\cdot,\cdot\rangle_{A^2_{-1}} = \langle\cdot,\cdot\rangle_{H^2}$. We will state many of our results in these terms, with the understanding that the $\alpha = -1$ "weighted Bergman space" always signifies the Hardy space.

For any analytic $\varphi : \mathbb{D} \to \mathbb{D}$, we will write $\|C_\varphi\|_{\mathcal{H}}$ to denote the norm of C_φ acting on a Hilbert space $\mathcal{H}$. While it is generally not easy to calculate $\|C_\varphi\|_{A^2_\alpha}$ explicitly, some concrete results are known – most notably in the case of the Hardy space H^2 (see [2], [3], [9], and [10]). Fortunately, it is not difficult to obtain sharp upper and lower bounds for the norm of C_φ. In particular, it is well known that

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\alpha+2} \le \|C_\varphi\|^2_{A^2_\alpha} \le \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+2} \tag{1}$$

for any $\alpha \ge -1$ (see Corollary 3.7 in [8] and Lemma 2.3 in [16]).

Reflecting on Equation (1), one might wonder whether there is some relationship between the quantities $\|C_\varphi\|_{A^2_\alpha}$ for different values of α. For example, considering $\alpha = 0$ and $\alpha = -1$, one might ask whether it is always the case that $\|C_\varphi\|_{A^2} = \|C_\varphi\|^2_{H^2}$. While this equality does hold for some maps, it is not true in general (see Section 4 of [4]). In this paper, we shall prove that $\|C_\varphi\|_{A^2} \le \|C_\varphi\|^2_{H^2}$ for all φ (see Corollary 5), answering a question posed by the authors of [4], and derive a collection of inequalities relating to the norms of C_φ on different spaces (see Theorem 4).

Before proceeding to our main results, we should mention a helpful fact relating to composition operators and reproducing kernel functions. Let C^*_φ denote the adjoint of C_φ on a particular space A^2_α; it is a simple exercise to show that $C^*_\varphi\big(K^\alpha_\lambda\big) = K^\alpha_{\varphi(\lambda)}$ for any λ in $\mathbb{D}$ (see Theorem 1.4 in [8]). This observation will provide exactly the information we need to compare the action of C_φ on different spaces.

2. Positive Semidefinite Matrices

Let $\Lambda = \{\lambda_m\}_{m=1}^{\infty}$, a sequence of distinct points in $\mathbb{D}$, be a set of uniqueness for the collection of analytic functions on $\mathbb{D}$. In other words, the zero function is the only analytic f with $f(\lambda_m) = 0$ for all m. (For example, Λ could be any sequence with a limit point inside $\mathbb{D}$.) The span of the kernel functions $\left\{K_{\lambda_m}^{\alpha}\right\}_{m=1}^{\infty}$ is dense in every space A_α^2, since any function orthogonal to each $K_{\lambda_m}^{\alpha}$ must be identically 0. For the duration of this paper, we will assume that such a sequence Λ has been fixed.

Consider an analytic map $\varphi : \mathbb{D} \to \mathbb{D}$. For a positive constant ν, a natural number n, and real number $\alpha \geq -1$, define the $n \times n$ matrix

$$M(\nu, n, \alpha) = \left[\frac{\nu^2}{(1 - \overline{\lambda_j}\lambda_i)^{\alpha+2}} - \frac{1}{(1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\alpha+2}}\right]_{i,j=1}^{n}.$$

Recall that an $n \times n$ matrix A is called *positive semidefinite* if $\langle Ac, c\rangle \geq 0$ for all c in $\mathbb{C}^n$, where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product. Any such matrix must necessarily be self-adjoint. We often write $A \geq 0$ to denote A being positive semidefinite; for self-adjoint matrices A and B, we write $A \geq B$ to denote $A - B$ being positive semidefinite. The following proposition relates $\|C_\varphi\|_{A_\alpha^2}$ to the positive semidefiniteness of $M(\nu, n, \alpha)$.

Proposition 1. *Let φ be an analytic self-map of $\mathbb{D}$ and ν be a positive constant. Then, for any $\alpha \geq -1$, the matrix $M(\nu, n, \alpha)$ is positive semidefinite for all natural numbers n if and only if $\|C_\varphi\|_{A_\alpha^2} \leq \nu$.*

Proof. Assume first that $\|C_\varphi\|_{A_\alpha^2} \leq \nu$, from which it follows that $\|C_\varphi^*\|_{A_\alpha^2} \leq \nu$. In other words,

$$\|C_\varphi^*(f)\|_{A_\alpha^2}^2 \leq \nu^2 \|f\|_{A_\alpha^2}^2 \tag{2}$$

for all f in A_α^2. Let n be any natural number and $c_1, \ldots, c_n$ be complex numbers, and take $f = c_1 K_{\lambda_1}^{\alpha} + \ldots + c_n K_{\lambda_n}^{\alpha}$. If we substitute this function into inequality (2), remembering that $C_\varphi^*(K_\lambda^\alpha) = K_{\varphi(\lambda)}^{\alpha}$, we obtain

$$\|c_1 K_{\varphi(\lambda_1)}^{\alpha} + \ldots + c_n K_{\varphi(\lambda_n)}^{\alpha}\|_{A_\alpha^2}^2 \leq \nu^2 \|c_1 K_{\lambda_1}^{\alpha} + \ldots + c_n K_{\lambda_n}^{\alpha}\|_{A_\alpha^2}^2,$$

from which it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j \left\langle K_{\varphi(\lambda_j)}^{\alpha}, K_{\varphi(\lambda_i)}^{\alpha} \right\rangle_{A_\alpha^2} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \nu^2 \overline{c_i} c_j \left\langle K_{\lambda_j}^{\alpha}, K_{\lambda_i}^{\alpha} \right\rangle_{A_\alpha^2},$$

and thus

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \overline{c_i} c_j \left(\frac{\nu^2}{(1 - \overline{\lambda_j}\lambda_i)^{\alpha+2}} - \frac{1}{(1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\alpha+2}} \right) \geq 0. \tag{3}$$

Inequality (3) is exactly the statement that $M(\nu, n, \alpha)$ is positive semidefinite.

For the converse, assume that $M(\nu, n, \alpha)$ is positive semidefinite for all natural numbers n. Hence inequality (3) holds for all n, which in turn implies that

$$\|c_1 K^\alpha_{\varphi(\lambda_1)} + \ldots + c_n K^\alpha_{\varphi(\lambda_n)}\|^2_{A^2_\alpha} \leq \nu^2 \|c_1 K^\alpha_{\lambda_1} + \ldots + c_n K^\alpha_{\lambda_n}\|^2_{A^2_\alpha} \tag{4}$$

for any n and any complex constants $c_1, \ldots, c_n$. Now let f be an arbitrary element of A^2_α. Since Λ is a set of uniqueness, the span of $\left\{K^\alpha_{\lambda_n}\right\}_{n=1}^\infty$ is dense in A^2_α. Hence there exists a sequence $\{f_m\}_{m=1}^\infty$ that converges to f in norm, where each f_m is a finite linear combination of these kernel functions. Line (4) implies that $\|C^*_\varphi(f_m)\|^2_{A^2_\alpha} \leq \nu^2 \|f_m\|^2_{A^2_\alpha}$ for all m. Letting m go to infinity, we see that $\|C^*_\varphi(f)\|^2_{A^2_\alpha} \leq \nu^2 \|f\|^2_{A^2_\alpha}$, from which it follows that $\|C_\varphi\|_{A^2_\alpha} = \|C^*_\varphi\|_{A^2_\alpha} \leq \nu$. □

In other words, Proposition 1 states that $\|C_\varphi\|_{A^2_\alpha} \leq \nu$ exactly when

$$\kappa(\lambda, z) = \frac{\nu^2}{(1 - \overline{\lambda} z)^{\alpha+2}} - \frac{1}{(1 - \overline{\varphi(\lambda)}\varphi(z))^{\alpha+2}}$$

is a positive semidefinite kernel on the unit disk.

Before proceeding to our main results, we need the following lemma relating to positive semidefinite matrices.

Lemma 2. *Let n be a natural number and $\lambda_1, \ldots, \lambda_n$ be a finite collection of (not necessarily distinct) points in $\mathbb{D}$. Any matrix of the form*

$$M = \left[\frac{1}{(1 - \overline{\lambda_j}\lambda_i)^\rho}\right]_{i,j=1}^n,$$

for any real number $\rho \geq 1$, must be positive semidefinite.

Proof. Let $\alpha = \rho - 2$, so that $\alpha \geq -1$. Taking $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, we see that

$$\langle Mc, c\rangle = \sum_{i=1}^n \sum_{j=1}^n \frac{\overline{c_i} c_j}{(1 - \overline{\lambda_j}\lambda_i)^{\alpha+2}} = \left\langle \sum_{j=1}^n c_j K^\alpha_{\lambda_j}, \sum_{i=1}^n c_i K^\alpha_{\lambda_i} \right\rangle_{A^2_\alpha} \geq 0,$$

from which our assertion follows. □

As a consequence of Lemma 2, we see that any matrix of the form

$$\left[\frac{1}{(1 - \overline{\varphi(\lambda_j)}\varphi(\lambda_i))^\rho}\right]_{i,j=1}^n,$$

where φ is a self-map of $\mathbb{D}$, must also be positive semidefinite.

3. Norm Inequalities

The proof of our major theorem relies heavily on the use of Schur products. Recall that, for any two $n \times n$ matrices $A = [a_{i,j}]_{i,j=1}^n$ and $B = [b_{i,j}]_{i,j=1}^n$, the *Schur* (or *Hadamard*) *product* $A \circ B$ is defined by the rule $A \circ B = [a_{i,j} b_{i,j}]_{i,j=1}^n$. In other words, the Schur product is obtained by entrywise multiplication. A proof of the following result appears in Section 7.5 of [12].

Proposition 3 (Schur Product Theorem). *If A and B are $n \times n$ positive semidefinite matrices, then $A \circ B$ is also positive semidefinite.*

We are now in position to state our main result, a theorem that allows us to compare the norms of C_φ on certain spaces.

Theorem 4. *Take $\beta \geq \alpha \geq -1$ and let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^\gamma_{A^2_\alpha} \tag{5}$$

whenever the quantity $\gamma = (\beta + 2)/(\alpha + 2)$ is an integer.

Proof. Assume that $\gamma = (\beta+2)/(\alpha+2)$ is an integer. Fix a natural number n and let $i, j \in \{1, \dots, n\}$. A difference of powers factorization shows that

$$\begin{aligned}\frac{\|C_\varphi\|^{2\gamma}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{\beta+2}} - \frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\beta+2}} &= \\ \left(\frac{\|C_\varphi\|^{2}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{\alpha+2}}\right) \\ \cdot \left(\sum_{k=0}^{\gamma-1} \frac{\|C_\varphi\|^{2k}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{(\alpha+2)k}(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{(\alpha+2)(\gamma-1-k)}}\right).\end{aligned}$$

Since the preceding equation holds for all i and j, we obtain the following matrix equation:

$$\begin{aligned}&M(\|C_\varphi\|^\gamma_{A^2_\alpha}, n, \beta) = \\ &M(\|C_\varphi\|_{A^2_\alpha}, n, \alpha) \circ \sum_{k=0}^{\gamma-1} \left[\frac{\|C_\varphi\|^{2k}_{A^2_\alpha}}{(1-\overline{\lambda_j}\lambda_i)^{(\alpha+2)k}(1-\overline{\varphi(\lambda_j)}\varphi(\lambda_i))^{(\alpha+2)(\gamma-1-k)}}\right]_{i,j=1}^{n}\end{aligned} \tag{6}$$

where $\circ$ denotes the Schur product. The matrix $M(\|C_\varphi\|_{A^2_\alpha}, n, \alpha)$ is positive semidefinite by Proposition 1. Lemma 2, together with the the Schur Product Theorem, dictates that every term in the matrix sum on the right-hand side of (6) is positive semidefinite, so the sum itself is positive semidefinite. Therefore the Schur Product Theorem shows that $M(\|C_\varphi\|^\gamma_{A^2_\alpha}, n, \beta)$ must also be positive semidefinite. Since this assertion holds for every natural number n, Proposition 1 shows that $\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^\gamma_{A^2_\alpha}$. □

Taking $\alpha = -1$ and $\alpha = 0$, we obtain the following corollaries.

Corollary 5. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^{\beta+2}_{H^2}$$

whenever β is a non-negative integer. In particular, $\|C_\varphi\|_{A^2} \leq \|C_\varphi\|^2_{H^2}$.

Corollary 6. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|_{A^2}^{(\beta+2)/2}$$

whenever β is a positive even integer.

Corollary 5 is particularly useful since, as we have already mentioned, more is known about the norm of C_φ on H^2 than on any other space. Hence any result pertaining to $\|C_\varphi\|_{H^2}$ can be translated into an upper bound for $\|C_\varphi\|_{A^2_\beta}$. The significance of Corollary 6 will become apparent in the next section.

There are certainly instances of analytic $\varphi : \mathbb{D} \to \mathbb{D}$ for which there is equality in line (5) for all α and β. If $\varphi(0) = 0$, for example, then line (1) shows that $\|C_\varphi\|_{A^2_\alpha} = 1$ for all α. A slightly less trivial class of examples are the maps $\varphi(z) = sz + t$, where s and t are complex numbers with $|s| + |t| \leq 1$. Combining results of Cowen [5] and Hurst [14], we see that

$$\|C_\varphi\|_{A^2_\alpha} = \left(\frac{2}{1 + |s|^2 - |t|^2 + \sqrt{(1 - |s|^2 + |t|^2)^2 - 4|t|^2}}\right)^{(\alpha+2)/2}$$

for any $\alpha \geq -1$. On the other hand, as noted in [4], there are cases where the inequality in (5) is strict, at least for some choices of α and β. For example, if φ is a non-univalent inner function that does not fix the origin, Theorem 3.3 in [4] shows that $\|C_\varphi\|_{A^2_\beta} < \|C_\varphi\|_{H^2}^{\beta+2}$ for all $\beta > -1$.

4. Open Questions

The major unanswered question, of course, is whether the conclusion of Theorem 4 still holds when the quantity γ is not an integer. In particular, one might wonder whether Corollary 6 can be extended to odd values of β.

The proof of Theorem 4 cannot be automatically extended to non-integer values of γ, since the Schur Product Theorem cannot be generalized to non-integer entrywise powers. If $A = [a_{i,j}]_{i,j=1}^n$ is self-adjoint, the *entrywise* (or *Hadamard*) *power* $A^{\circ,\gamma}$ is defined by the rule $A^{\circ,\gamma} = \left[a_{i,j}^\gamma\right]_{i,j=1}^n$, where the arguments of the entries of A are chosen consistently so that all of the matrix powers are self-adjoint. It turns out that the condition $A \geq 0$ does not imply that $A^{\circ,\gamma} \geq 0$ for non-integer values of γ. (If a matrix A does have the special property that $A^{\circ,\gamma} \geq 0$ for all $\gamma \geq 0$, then A is called *infinitely divisible*. A necessary and sufficient condition for this property is discussed in Section 6.3 of [13].) The proof of Theorem 4 essentially involves using the Schur Product Theorem to show that $A \geq B \geq 0$ implies $A^{\circ,k} \geq B^{\circ,k}$ whenever k is a positive integer. Little seems to be known, however, about conditions on A and B which would guarantee that $A \geq B \geq 0$ implies $A^{\circ,\gamma} \geq B^{\circ,\gamma}$ for all $\gamma \geq 1$. Such conditions could help determine to what extent Theorem 4 can be generalized.

Taking a different point of view, one might try to "fill in the gaps" of Theorem 4 using some sort of interpolation argument (such as Theorem 1.1 in [15]). While such techniques initially appear promising, they generally involve working with Hilbert spaces that have equivalent norms to the spaces in which we are interested. Hence such an approach cannot be applied to any question that deals with the precise value of an operator norm.

It might be helpful to recast this question in terms of the relationship between the norm of a composition operator and the property of cosubnormality (that is, the adjoint of the operator being subnormal). Based on the scant evidence we have (see [2] and [3]), one might conjecture that, for any univalent φ with Denjoy–Wolff point on $\partial\mathbb{D}$, the norm of C_φ equals its spectral radius on A^2_α if and only if C_φ is cosubnormal on that space. If that conjecture were accurate, then Corollary 6 would not hold for odd values of β.

In particular, consider the maps of the form

$$\varphi(z) = \frac{(r+s)z + 1 - s}{r(1-s)z + 1 + sr} \tag{7}$$

for $-1 \le r \le 1$ and $0 < s < 1$, a class introduced by Cowen and Kriete [7]. Richman [16] showed that C_φ is cosubnormal on A^2 precisely when $-1/7 \le r \le 1$. On the other hand, he showed in [17] that C_φ is cosubnormal on A^2_1 if and only if $0 \le r \le 1$. Take, for example,

$$\varphi(z) = \frac{7}{8-z},$$

which corresponds to (7) with $r = -1/7$ and $s = 1/7$. We know that C_φ is cosubnormal on A^2, which means that its norm on A^2 is equal to its spectral radius, which is $\varphi'(1)^{-1} = 7$. On the other hand, C_φ is not cosubnormal on A^2_1, so it is *possible* that its norm on that space might exceed its spectral radius, which is $7^{3/2}$. If that were the case, then Corollary 6 – and hence Theorem 4 – would not be valid for intermediate spaces. We have attempted (in the spirit of [1]) to show that $\|C_\varphi\|_{A^2_1} > 7^{3/2}$ through a variety of numerical calculations, all of which have been inconclusive.

The following result, a sort of "cousin" to our Theorem 4, may also be relevant to the question at hand:

Theorem 7 (Cowen [6]**).** *Take $\beta \ge \alpha \ge -1$ and let φ be an analytic self-map of $\mathbb{D}$. Suppose that $\gamma = (\beta+2)/(\alpha+2)$ is an integer. If C_φ is cosubnormal on A^2_α, then it is also cosubnormal on A^2_β.*

Cowen only stated this result for $\alpha = -1$, but an identical argument works for $\alpha > -1$. The proof makes use of the Schur Product Theorem in a similar fashion to that of Theorem 4. Moreover, we know that the result does *not* hold for intermediate spaces. For example,

$$\varphi(z) = \frac{7}{8-z},$$

induces a cosubnormal composition operator on A^2, and hence on A^2_2, but not on the space A^2_1.

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