Norm Inequalities for Composition Operators on Hardy and Weighted Bergman Spaces

Christopher Hammond
Connecticut College

(joint work with Linda J. Patton)

July 24, 2008

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The *Hardy space* H^2 consists of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right) \right|^2 \frac{d\theta}{2\pi} < \infty.$$

For $\alpha > -1$, the *weighted Bergman space* A_α^2 consists of all analytic f on $\mathbb{D}$ such that

$$\|f\|_{A_\alpha^2}^2 = \int_{\mathbb{D}} |f(z)|^2 (\alpha + 1)(1 - |z|^2)^\alpha \, dA(z) < \infty,$$

where dA signifies normalized area measure on $\mathbb{D}$.

All of these spaces are *reproducing kernel Hilbert spaces*.

For every point λ in $\mathbb{D}$ there is a unique K_λ in H^2 such that

$$\langle f, K_\lambda \rangle = f(\lambda)$$

for all f in H^2.

It is not difficult to show that

$$K_\lambda = \frac{1}{1 - \overline{\lambda} z}.$$

Furthermore, the span of the kernel functions is dense in H^2.

Likewise, for any $\alpha > -1$, one can show that

$$\langle f, K_\lambda^\alpha \rangle = f(\lambda)$$

for any f in A_α^2 and λ in $\mathbb{D}$, where

$$K_\lambda^\alpha(z) = \frac{1}{(1-\overline{\lambda}z)^{\alpha+2}}.$$

For this reason, we shall refer to H^2 as A_{-1}^2.

Let φ be an analytic map from $\mathbb{D}$ into itself.

We define the *composition operator* C_φ on H^2 by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Every such operator is bounded on any space A^2_α for $\alpha \geq -1$.

Moreover,

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{\alpha+2} \leq \|C_\varphi\|^2_{A^2_\alpha} \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{\alpha+2}$$

Question: Is there any relationship between the norm of C_φ on different spaces?

For instance, does $\|C_\varphi\|_{A^2_\beta}^{1/(\beta+2)} = \|C_\varphi\|_{A^2_\alpha}^{1/(\alpha+2)}$?

In particular, does $\|C_\varphi\|_{A^2}^{1/2} = \|C_\varphi\|_{H^2}$?

Examples:

(1) $\varphi(0) = 0 \quad \Rightarrow \quad \|C_\varphi\|_{A^2_\alpha} = 1.$

(2) $\varphi(z) = sz + t \quad \Rightarrow$

$$\|C_\varphi\|_{A^2_\alpha} = \left(\frac{2}{1+|s|^2-|t|^2+\sqrt{(1-|s|^2+|t|^2)^2-4|t|^2}}\right)^{(\alpha+2)/2}$$

Nevertheless, there are examples where such an equality does not hold.

Examples [Carswell, H 2006]:

The map

$$\varphi(z) = \frac{7}{8 - z}$$

has the property that $\|C_\varphi\|_{A^2} = 7$, whereas $\|C_\varphi\|_{H^2} > \sqrt{7}$.

Likewise, for any non-univalent inner function φ with $\varphi(0) \neq 0$, one can demonstrate that $\|C_\varphi\|_{A^2}^{1/2} < \|C_\varphi\|_{H^2}$.

Our main result:

Theorem 1. *Take* $\beta \geq \alpha \geq -1$ *and let* φ *be an analytic self-map of* $\mathbb{D}$. *Then*

$$\|C_\varphi\|_{A^2_\beta} \leq \|C_\varphi\|^\gamma_{A^2_\alpha}$$

whenever the quantity $\gamma = (\beta + 2)/(\alpha + 2)$ *is an integer.*

Take $\alpha = -1$:

Corollary 2. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \le \|C_\varphi\|_{H^2}^{\beta+2}$$

whenever β is a non-negative integer. In particular, $\|C_\varphi\|_{A^2} \le \|C_\varphi\|_{H^2}^2$.

Take $\alpha = 0$:

Corollary 3. *Let φ be an analytic self-map of $\mathbb{D}$. Then*

$$\|C_\varphi\|_{A^2_\beta} \le \|C_\varphi\|_{A^2}^{(\beta+2)/2}$$

whenever β is a positive even integer.

Sketch of the proof of Theorem 1:

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a fixed sequence of points in $\mathbb{D}$ with a limit point in $\mathbb{D}$.

For a positive constant ν, a natural number n, and real number $\alpha \geq -1$, let $M(\nu, n, \alpha)$ be the $n \times n$ matrix

$$\left[\frac{\nu^2}{(1-\overline{\lambda_i}\lambda_j)^{\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda_i)}\varphi(\lambda_j))^{\alpha+2}}\right]_{i,j=1}^{n}.$$

Recall that an $n \times n$ matrix A is called *positive semidefinite* if $\langle Ac, c \rangle \geq 0$ for all c in $\mathbb{C}^n$.

Proposition 4. *Let φ be an analytic self-map of $\mathbb{D}$ and ν be a positive constant. Then, for any $\alpha \geq -1$, the matrix $M(\nu, n, \alpha)$ is positive semidefinite for all natural numbers n if and only if $\|C_\varphi\|_{A^2_\alpha} \leq \nu$.*

Hence $M(\|C_\varphi\|_{A^2_\alpha}, n, \alpha)$ is positive semidefinite for all n.

Our goal is to show that $M(\|C_\varphi\|^\gamma_{A^2_\alpha}, n, \beta)$ is positive semidefinite for all n whenever $\gamma = (\beta + 2)/(\alpha + 2)$ is an integer.

For any two matrices $A = [a_{i,j}]$ and $B = [b_{i,j}]$, we define the *Schur product* $A \circ B = [a_{i,j} b_{i,j}]$.

Schur Product Theorem. If A and B are $n \times n$ positive semidefinite matrices, then their Schur product $A \circ B$ is also positive semidefinite.

We would like to write

$$M(\|C_\varphi\|^\gamma_{A^2_\alpha}, n, \beta) = M(\|C_\varphi\|_{A^2_\alpha}, n, \alpha) \circ N,$$

where N is positive semidefinite.

Observe that

$$\frac{\|C_\varphi\|_{A^2_\alpha}^{2\gamma}}{(1-\overline{\lambda_i}\lambda_j)^{\beta+2}} - \frac{1}{(1-\overline{\varphi(\lambda_i)}\varphi(\lambda_j))^{\beta+2}} =$$

$$\left(\frac{\|C_\varphi\|_{A^2_\alpha}^{2}}{(1-\overline{\lambda_i}\lambda_j)^{\alpha+2}} - \frac{1}{(1-\overline{\varphi(\lambda_i)}\varphi(\lambda_j))^{\alpha+2}}\right)$$
$$\cdot\left(\sum_{k=0}^{\gamma-1} \frac{\|C_\varphi\|_{A^2_\alpha}^{2k}}{(1-\overline{\lambda_i}\lambda_j)^{(\alpha+2)k}(1-\overline{\varphi(\lambda_i)}\varphi(\lambda_j))^{(\alpha+2)(\gamma-1-k)}}\right)$$

for all $1 \leq i, j \leq n$.

Question: Does Theorem 1 hold for values of α and β such that $\gamma = (\beta + 2)/(\alpha + 2)$ is not an integer?

For example, could there be some φ such that $\|C_\varphi\|_{A^2_1} > \|C_\varphi\|^{3/2}_{A^2}$?

At this point, the answer is unknown.

A related result:

Theorem [Cowen 1992]. Take $\beta \geq \alpha \geq -1$ and let $\varphi : \mathbb{D} \to \mathbb{D}$. Suppose $\gamma = (\beta+2)/(\alpha+2)$ is an integer. If C_φ is cosubnormal on A^2_α, then it is also cosubnormal on A^2_β.

This result does *not* hold for intermediate spaces. For example,

$$\varphi(z) = \frac{7}{8-z},$$

induces a cosubnormal composition operator on A^2, and hence on A^2_2, but not on A^2_1.

The example

$$\varphi(z) = \frac{7}{8 - z},$$

might provide an example where Theorem 1 fails for intermediate spaces.

We know that $\|C_\varphi\|_{A^2} = 7$ and $\|C_\varphi\|_{A^2_2} = 49$.

It is *possible* that $\|C_\varphi\|_{A^2_1} > 7^{3/2}$.