Zeros of Hypergeometric Functions and the Norm of a Composition Operator

Christopher Hammond
Connecticut College

October 7, 2005

Let H^2 denote the Hardy space on the disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. We define the *composition operator* C_φ on H^2 by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Every such φ induces a bounded composition operator on H^2.

Moreover, we know that

$$\frac{1}{1-|\varphi(0)|^2} \leq \|C_\varphi\|^2 \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}$$

for any $\varphi : \mathbb{D} \to \mathbb{D}$.

If $\varphi(0) = 0$, then $\|C_\varphi\| = 1$. Otherwise, it is difficult to determine $\|C_\varphi\|$ explicitly.

One common strategy for determining $\|C_\varphi\|$ is to study the spectrum of $C_\varphi^* C_\varphi$.

Justification:

Let T be a bounded operator on a Hilbert space $\mathcal{H}$. We know that:

- The spectral radius of T^*T equals $\|T^*T\| = \|T\|^2$.

- Let $h \in \mathcal{H}$; then $\|T(h)\| = \|T\| \, \|h\|$ if and only if $(T^*T)(h) = \|T\|^2 \, h$.

- If $\|T\|_e < \|T\|$, then T is norm-attaining.

Let

$$\varphi(z) = \frac{az + b}{cz + d}$$

be a self-map of $\mathbb{D}$, with $ad - bc \neq 0$. Then

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}.$$

is also a self-map of $\mathbb{D}$.

Cowen [1988] showed that $C_\varphi^* = T_\gamma C_\sigma T_\eta^*$, where

$$\gamma(z) = \frac{1}{-\overline{b}z + \overline{d}} \text{ and } \eta(z) = cz + d.$$

Cowen's adjoint formula shows that

$$\left(C_\varphi^* C_\varphi f\right)(z) = \psi(z) f(\tau(z)) + \chi(z) f(\varphi(0)),$$

where

$$\begin{aligned}
\tau(z) &= \varphi(\sigma(z)), \\
\psi(z) &= \frac{(\overline{ad} - \overline{bc})z}{(\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})}, \text{ and} \\
\chi(z) &= \frac{\overline{c}}{-\overline{a}z + \overline{c}}.
\end{aligned}$$

This representation holds for all for z except for $\sigma^{-1}(0) = \overline{c}/\overline{a}$.

Using this representation, it is possible to obtain some results about $\|C_\varphi\|$:

Theorem 1. (Bourdon, Fry, H, Spofford [2004]) *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a non-automorphic linear fractional map that fixes the point* 1*. If* λ *is an eigenvalue of* $C_\varphi^* C_\varphi$ *with* $\lambda > \|C_\varphi\|_e^2$*, then* λ *is a solution to the equation*

$$\sum_{k=0}^{\infty} \chi(\tau^{[k]}(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau^{[m]}(\varphi(0))) \right] \left(\frac{1}{\lambda} \right)^{k+1} = 1.$$

Conversely, any complex number $|\lambda| > \|C_\varphi\|_e^2$ *that is a solution to this equation is an eigenvalue for* $C_\varphi^* C_\varphi$*.*

For a, b, and c in $\mathbb{C}$, we define the *hypergeometric series*

$$ {}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, $$

where $(\cdot)_k$ denotes *Pochhammer's symbol*:

$$ (\zeta)_k := \begin{cases} 1, & k=0 \\ \zeta(\zeta+1)\ldots(\zeta+k-1), & k=1,2,3,\ldots . \end{cases} $$

Any non-automorphic linear fractional map $\varphi : \mathbb{D} \to \mathbb{D}$ that fixes 1 can be written

$$\varphi(z) = \frac{(\overline{\beta} - 1)z + \alpha + 1}{\overline{\alpha} z + \beta},$$

where $\overline{\alpha} + \beta > 0$ and $\beta - \alpha - 1 > 0$.

Basor and Retsek [2005] showed that

$$\sum_{k=0}^{\infty} \chi(\tau^{[k]}(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau^{[m]}(\varphi(0))) \right] z^{k+1}$$

equals $1 - {}_2F_1(\alpha, \beta; \delta; z/q)$, where

$$\delta = \overline{\alpha} + \beta \text{ and } q = \varphi'(1) = \frac{\beta - \alpha - 1}{\overline{\alpha} + \beta}.$$

In other words, $\lambda > 1/q = \|C_\varphi\|_e^2$ is an eigenvalue for $C_\varphi^* C_\varphi$ if and only if $(q\lambda)^{-1} = \|C_\varphi\|_e^2\,\lambda^{-1}$ in $(0,1)$ is a zero of ${}_2F_1(\alpha,\beta;\delta;z)$.

Basor and Retsek showed that $C_\varphi^* C_\varphi$ has at least one such eigenvalue, except in the case where α is real and positive.

Question: How many eigenvalues does $C_\varphi^* C_\varphi$ have that are greater than $\|C_\varphi\|_e^2$?

Van Vleck [1902] determined the number of zeros of ${}_2F_1(a, b; c; z)$ when a, b, and c are real. For example, if $c > 1$ the series has

$$E\left(\frac{|a-b| - |1-c| - |c-a-b| + 1}{2}\right)$$

zeros in $(0, 1)$.

Here $E(\cdot)$ denotes *Klein's symbol*:

$$E(u) := \begin{cases} 0, & u \leq 0 \\ \lfloor u \rfloor, & u > 0,\ u \text{ not an integer} \\ u-1, & u = 1, 2, 3, \ldots \end{cases}$$

Using Van Vleck's results, we can show that ${}_2F_1(\alpha, \beta; \delta; z)$ has $E(-\alpha + 1)$ zeros in $(0, 1)$ when α is real.

In other words, we have obtained the following result:

Proposition 2. *Let α and β be real, with $\alpha + \beta > 0$ and $\beta - \alpha - 1 > 0$; consider the map*

$$\varphi(z) = \frac{(\beta - 1)z + \alpha + 1}{\alpha z + \beta}.$$

The operator $C_\varphi^* C_\varphi$ has exactly $E(-\alpha+1)$ eigenvalues greater than $\|C_\varphi\|_e^2$.

This result has an interesting geometric interpretation. Note that the point $\tau(0) = \varphi(\sigma(0))$ is the center of the disk $\varphi(\mathbb{D})$.

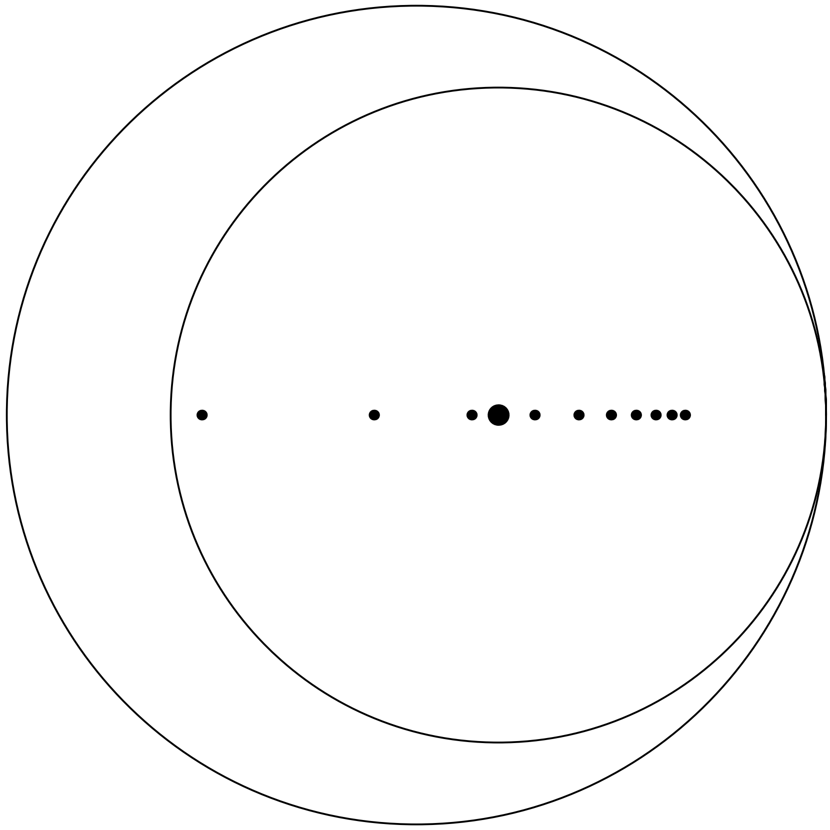
Corollary 3. *Consider the map*

$$\varphi(z) = \frac{(\beta - 1)z + \alpha + 1}{\alpha z + \beta}.$$

The operator $C_\varphi^* C_\varphi$ has exactly m eigenvalues greater than $\|C_\varphi\|_e^2$, where m is the smallest non-negative integer such that $\tau^{[m]}(\varphi(0)) \geq \tau(0)$.

$$\text{Example: } \varphi(z) = \frac{13z - 11}{-19z + 21}.$$

(Note that $\alpha = -19/8$, so $E(-\alpha + 1) = 3$.)



It is not difficult to determine the spectrum of $C_\varphi^* C_\varphi$ entirely:

Theorem 4. *Consider the map*

$$\varphi(z) = \frac{(\beta - 1)z + \alpha + 1}{\alpha z + \beta}.$$

The spectrum of the operator $C_\varphi^* C_\varphi$ *is precisely*

$$\left[0, \|C_\varphi\|_e^2\right] \cup \{\lambda_k\}_{k=1}^m ,$$

where $m = E(-\alpha + 1)$ *and* $\lambda_1, \lambda_2, \ldots, \lambda_m$ *are distinct eigenvalues greater than* $\|C_\varphi\|_e^2$.

Theorem 5. *Let* α *and* β *be complex numbers, with* $\delta = \overline{\alpha} + \beta > 0$ *and* $\beta - \alpha - 1 > 0$*. All of the zeros of the hypergeometric series* ${}_2F_1(\alpha, \beta; \delta; z)$ *within* $\mathbb{D}$ *must lie on the positive real axis. Moreover, the smallest such zero must be greater than or equal to*

$$\frac{(\overline{\alpha} + \beta)(|\beta| - |\alpha + 1|)}{(\beta - \alpha - 1)(|\beta| + |\alpha + 1|)}$$

and less than or equal to

$$\frac{(\overline{\alpha} + \beta)(|\beta|^2 - |\alpha + 1|^2)}{(\beta - \alpha - 1)|\beta|^2}.$$