Composition Operators on Spaces of Analytic Functions

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Two old problems in Analysis:

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let φ be an analytic map that takes $\mathbb{D}$ into itself.

First Problem: Describe the function-theoretic properties of such maps.

Denjoy–Wolff Theorem [1926]
Suppose φ is not the identity and not an elliptic automorphism. Then φ has a unique fixed point w in $\overline{\mathbb{D}}$ such that the iterates $\varphi^{[n]}$ converge to w uniformly on compact subsets of $\mathbb{D}$.

Second problem: Understand and describe the bounded operators on a Hilbert space (in other words, the continuous linear transformations on a complete inner product space).

One way to attack this problem is to study the behavior of certain classes of operators.

For example, let $\{e_n\}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. We could define an operator $T : \mathcal{H} \to \mathcal{H}$ in terms of this basis.

We could consider a *diagonal operator*:

$$T(e_n) = \alpha_n e_n,$$

where $\sup |\alpha_n| < \infty$.

We could also consider a *weighted shift*:

$$T(e_n) = \alpha_n e_{n+1},$$

where $\sup |\alpha_n| < \infty$.

We could also consider operators $T : \mathcal{H} \to \mathcal{H}$ with certain symmetry properties.

Recall that the *adjoint* T^* is the (unique) bounded operator on $\mathcal{H}$ such that

$$\langle T(h), k \rangle = \langle h, T^*(k) \rangle$$

for all h and k in $\mathcal{H}$.

An operator $T : \mathcal{H} \to \mathcal{H}$ is *self-adjoint* (or *Hermitian*) if $T^* = T$.

An operator $T : \mathcal{H} \to \mathcal{H}$ is *normal* if $T^*T = TT^*$.

Another symmetry property.

Let $C : \mathcal{H} \to \mathcal{H}$ be an antilinear involutive isometry; that is:

- $C(h+k) = C(h) + C(k)$ for h and k in $\mathcal{H}$,
- $C(\alpha h) = \overline{\alpha} C(h)$ for h in $\mathcal{H}$ and α in $\mathbb{C}$,
- $C(C(h)) = h$ for all h in $\mathcal{H}$,
- $\langle h, k \rangle = \langle C(k), C(h) \rangle$ for h and k in $\mathcal{H}$.

We say an operator $T : \mathcal{H} \to \mathcal{H}$ is *complex symmetric* if $CT = T^*C$ for some such C.

Let us consider a particular Hilbert space.

The *Hardy space* H^2 consists of all analytic functions $f(z) = \sum a_n z^n$ on $\mathbb{D}$ such that

$$\|f\| := \sqrt{\sum_{n=0}^{\infty} |a_n|^2} < \infty.$$

The inner product of two functions $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ is defined

$$\begin{aligned}\langle f, g\rangle &:= \sum_{n=0}^{\infty} a_n \overline{b_n} \\ &= \lim_{r \uparrow 1} \int_0^{2\pi} f\left(re^{i\theta}\right)\overline{g\left(re^{i\theta}\right)}\frac{d\theta}{2\pi}\end{aligned}$$

Note that the monomials $\{z^n\}$ constitute an orthonormal basis for H^2.

The linear transformation

$$(T(f))(z) = zf(z)$$

is simply the shift operator with respect to this basis.

An important class of functions in H^2: the *reproducing kernel functions*.

For any w in $\mathbb{D}$, define

$$K_w(z) = \frac{1}{1 - \overline{w}z} = \sum_{n=0}^{\infty} \overline{w}^n z^n.$$

Let $f = \sum a_n z^n$ be an arbitrary function in H^2. Note that

$$\langle f, K_w \rangle = \sum_{n=0}^{\infty} a_n \overline{\overline{w}^n} = \sum_{n=0}^{\infty} a_n w^n = f(w).$$

Note that the span of all kernel functions is dense in H^2.

Let φ be an analytic map from $\mathbb{D}$ into $\mathbb{D}$. We define the composition operator C_φ on H^2 by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Some natural questions:

- Is C_φ bounded on H^2?
- What is the spectrum of C_φ?
- What is the adjoint of C_φ?
- What is $\|C_\varphi\|$?

How do the answers to these questions relate to the function-theoretic properties of φ?

Boundedness

The Littlewood Subordination Theorem [1925] shows that C_φ is bounded on H^2. Furthermore,

$$\frac{1}{1-|\varphi(0)|^2} \leq \|C_\varphi\|^2 \leq \frac{1+|\varphi(0)|}{1-|\varphi(0)|}.$$

These bounds are sharp. There are certain φ (constant maps) where the norm equals the lower bound and others (inner functions) where the norm equals the upper bound.

Spectral Radius

Recall that the *spectrum* of $T : \mathcal{H} \to \mathcal{H}$ is defined as follows:

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not invertible}\}.$$

We write $r(T)$ to denote the *spectral radius* of T:

$$r(T) := \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

For any operator $T : \mathcal{H} \to \mathcal{H}$, we know that

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}.$$

We have enough information to calculate the spectral radius of $C_\varphi : H^2 \to H^2$.

Observe that $(C_\varphi)^n = C_{\varphi^{[n]}}$. Therefore

$$\left(\frac{1}{1-|\varphi^{[n]}(0)|^2}\right)^{1/2n} \le \|(C_\varphi)^n\|^{1/n} \le \left(\frac{1+|\varphi^{[n]}(0)|}{1-|\varphi^{[n]}(0)|}\right)^{1/2n}.$$

If the Denjoy–Wolff point w of φ lies inside $\mathbb{D}$, then we see that $r(C_\varphi) = 1$. If w lies on $\partial\mathbb{D}$, then $r(C_\varphi) = \varphi'(w)^{-1/2}$.

(If φ is an elliptic automorphism, $r(C_\varphi) = 1$.)

Adjoints

Can we find a concrete representation for C_φ^*?

A helpful observation:

$$\begin{aligned}\langle f, C_\varphi^*(K_w)\rangle &= \langle C_\varphi(f), K_w\rangle \\ &= \langle f \circ \varphi, K_w\rangle \\ &= f(\varphi(w)) \\ &= \langle f, K_{\varphi(w)}\rangle\end{aligned}$$

for all f in H^2, so $C_\varphi^*(K_w) = K_{\varphi(w)}$.

This fact allows us to compute C_φ^* when φ is linear fractional.

Theorem 1. [Cowen 1988]
Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ *has the form*

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. *Then* $C_\varphi^*(f)$ *is given by the formula*

$$\gamma(z)\left(\overline{c}\left(\frac{f(\sigma(z)) - f(0)}{\sigma(z)}\right) + \overline{d} f(\sigma(z))\right),$$

where

$$\gamma(z) = \frac{1}{-\overline{b}z + \overline{d}}$$

and

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}.$$

Proof. Consider the reproducing kernel function $K_w(z) = \frac{1}{1-\overline{w}z}$. Observe that

$$\gamma(z)\left(\overline{c}\left(\frac{K_w(\sigma(z)) - K_w(0)}{\sigma(z)}\right) + \overline{d}K_w(\sigma(z))\right)$$

$$= \gamma(z)\left(\frac{\overline{cw} + \overline{d}}{1 - \overline{w}\sigma(z)}\right)$$

$$= \left(\frac{1}{-\overline{b}z + \overline{d}}\right)\left(\frac{\overline{cw} + \overline{d}}{1 - \overline{w}\left(\frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}\right)}\right)$$

$$= \frac{\overline{cw} + \overline{d}}{-\overline{b}z + \overline{d} - \overline{wa}z + \overline{wc}}$$

$$= \frac{1}{1 - \overline{\left(\frac{aw+b}{cw+d}\right)}z}$$

$$= \frac{1}{1 - \overline{\varphi(w)}z} = K_{\varphi(w)}(z) = (C_\varphi^*(K_w))(z).$$

Cowen's formula can be rewritten

$$(C_\varphi^*(f))(z) = \frac{(\overline{ad} - \overline{bc})z}{(\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})} f(\sigma(z)) + \frac{\overline{c}f(0)}{\overline{c} - \overline{a}z}$$

$$= \frac{z\sigma'(z)}{\sigma(z)} f(\sigma(z)) + \frac{f(0)}{1 - (\overline{a}/\overline{c})z}.$$

A brand new result:

Theorem 2. [H, Moorhouse, Robbins 2007] *Suppose that φ is a rational map that takes $\mathbb{D}$ into itself. Then*

$$(C_\varphi^*(f))(z) = \sum \psi(z) f(\sigma(z)) + \frac{f(0)}{1 - \overline{\varphi(\infty)} z},$$

where

$$\sigma(z) = 1/\overline{\varphi^{-1}(1/\overline{z})},$$
$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)},$$
$$\varphi(\infty) = \lim_{|z|\to\infty} \varphi(z),$$

and the summation is taken over all branches of σ.

Examples

(1) Let $\varphi(z) = z^2$. Then

$$(C^*_\varphi(f))(z) = \frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}.$$

(2) Let

$$\varphi(z) = \frac{z^2 - 6z + 9}{z^2 - 10z + 13}.$$

Then $(C^*_\varphi(f))(z)$ equals

$$\sum_{j=1}^{2} \frac{(-1)^j\, 2z}{\sqrt{3-2z}(3z - 4 + (-1)^j \sqrt{3-2z})} \cdot f\left(\frac{3z - 5 + (-1)^j\, 2\sqrt{3-2z}}{9z - 13}\right) + \frac{f(0)}{1-z}.$$

Norms

Can we calculate $\|C_\varphi\|$ exactly?

For any operator $T : \mathcal{H} \to \mathcal{H}$, we know that

$$r(T^*T) = \|T^*T\| = \|T\|^2.$$

We can learn about $\|C_\varphi\|$ by studying the spectrum (particularly the eigenvalues) of $C_\varphi^* C_\varphi$.

A helpful observation:

Suppose f is an eigenvector for $C_\varphi^* C_\varphi$ corresponding to an eigenvalue λ. Then

$$\begin{aligned} f(\varphi(0)) &= \langle f \circ \varphi, K_0 \rangle \\ &= \langle C_\varphi(f), K_0 \rangle \\ &= \langle C_\varphi(f), C_\varphi(K_0) \rangle \\ &= \langle C_\varphi^* C_\varphi(f), K_0 \rangle \\ &= \langle \lambda f, K_0 \rangle \\ &= \lambda f(0). \end{aligned}$$

Example [H 2003]

Let $\varphi(z) = \dfrac{16z + 8}{19z + 32}$.

Cowen's adjoint formula for linear fractional maps shows that $(C_\varphi^* C_\varphi(f))(z)$ equals

$$\frac{45z}{(16z-19)(4-z)} f\left(\frac{64z-16}{16z+221}\right) + \frac{19}{19-16z} f(\varphi(0)).$$

If f is an eigenvector for $C_\varphi^* C_\varphi$ corresponding to $\lambda = \|C_\varphi\|^2$, we see that

$$\lambda f(\varphi(0)) = -\frac{1}{5} f(0) + \frac{19}{15} f(\varphi(0))$$

$$\lambda^2 f(0) = -\frac{1}{5} f(0) + \frac{19}{15} \lambda f(0).$$

Hence $\lambda = \|C_\varphi\|^2 = \dfrac{19 + \sqrt{181}}{30}$.

In general, it is possible to determine $\|C_\varphi\|$ when φ is linear fractional, at least in the case where $\varphi(1) = 1$.

Theorem 3. [Basor, Retsek 2006]
Suppose

$$\varphi(z) = \frac{(\overline{\beta} - 1)z + \alpha + 1}{\overline{\alpha} z + \beta}$$

takes $\mathbb{D}$ *into itself. Then* $1/\|C_\varphi\|^2$ *is the smallest zero of the hypergeometric function* ${}_2F_1(\alpha, \beta; \delta; z/q)$, *where*

$$\delta = \overline{\alpha} + \beta \text{ and } q = \varphi'(1) = \frac{\beta - \alpha - 1}{\overline{\alpha} + \beta}.$$

This result builds on [H 2003] and [Bourdon, Fry, H, Spofford 2004], and is studied further in [H 2006].

Some open questions:

- Can one find $\|C_\varphi\|$ when $\varphi : \mathbb{D} \to \mathbb{D}$ is a rational function?

- What can one say about $\|C_\varphi\|$ on other spaces?

Progress has been made recently by [Effinger-Dean, Johnson, Reed, Shapiro 2006] and by Patton [2007]. It might also be possible to use the new adjoint formula.

Component Structure

Let us consider $\mathcal{C}(H^2)$, the set of all composition operators, as a subset of $\mathcal{B}(H^2)$. What can we say about its component structure?

Theorem 4. [Shapiro, Sundberg 1990] (based on [Berkson 1981])
Let φ and ψ be distinct analytic self-maps of $\mathbb{D}$. Then

$$\|C_\varphi - C_\psi\|^2 \geq |E(\varphi)| + |E(\psi)|,$$

where $E(\varphi) = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$.

Corollary 5. *If $|E(\varphi)| > 0$, then C_φ is isolated in $\mathcal{C}(H^2)$.*

On the other hand, all the operators C_φ with $E(\varphi) = \varnothing$ belong to the same component of $\mathcal{C}(H^2)$.

Determining the exact component structure of $\mathcal{C}(H^2)$ is somewhat tricky.

Conjecture 6. [Shapiro, Sundberg] *The operators C_φ and C_ψ belong to the same component of $\mathcal{C}(H^2)$ if and only if $C_\varphi - C_\psi$ is compact.*

[Bourdon 2003] and [Moorhouse, Toews 2003] provide examples where C_φ and C_ψ belong to the same component of $\mathcal{C}(H^2)$, yet their difference is not compact.