Adjoints of Composition Operators with Rational Symbol

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Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The *Hardy space* H^2 is the Hilbert space consisting of all analytic functions $f(z) = \sum a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|_2 := \sqrt{\sum_{n=0}^{\infty} |a_n|^2} < \infty,$$

with

$$\begin{aligned}
\langle f, g \rangle : &= \sum_{n=0}^{\infty} a_n \overline{b_n} \\
&= \lim_{r \uparrow 1} \int_0^{2\pi} f\left(re^{i\theta}\right) \overline{g\left(re^{i\theta}\right)} \frac{d\theta}{2\pi} \\
&= \int_0^{2\pi} f\left(e^{i\theta}\right) \overline{g\left(e^{i\theta}\right)} \frac{d\theta}{2\pi} \\
&= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \overline{g(\zeta)} \frac{d\zeta}{\zeta}.
\end{aligned}$$

Recall that H^2 is a *reproducing kernel Hilbert space*. That is, for any w in $\mathbb{D}$, there is a function K_w in H^2 such that

$$\langle f, K_w \rangle = f(w)$$

for all f in H^2.

It is easy to show that

$$K_w(z) = \frac{1}{1 - \overline{w}z}.$$

Note that the span of the kernel functions is dense in H^2.

Let φ be an analytic map from $\mathbb{D}$ into itself.

We define the *composition operator* C_φ on H^2 by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Every such operator is bounded on the Hardy space.

We would like to find a concrete representation for the adjoint C_φ^*.

A helpful observation:

$$\begin{aligned}\langle f, C_\varphi^*(K_w)\rangle &= \langle C_\varphi(f), K_w\rangle \\ &= \langle f \circ \varphi, K_w\rangle \\ &= f(\varphi(w)) \\ &= \langle f, K_{\varphi(w)}\rangle\end{aligned}$$

for all f in H^2, so $C_\varphi^*(K_w) = K_{\varphi(w)}$.

Another useful observation:

Note that

$$\begin{aligned}(C_\varphi^* f)(w) &= \langle C_\varphi^*(f), K_w\rangle \\ &= \langle f, C_\varphi(K_w)\rangle \\ &= \langle f, K_w \circ \varphi\rangle \\ &= \int_0^{2\pi} \frac{f\left(e^{i\theta}\right)}{1 - \overline{\varphi\left(e^{i\theta}\right)}w}\frac{d\theta}{2\pi}\end{aligned}$$

for all w in $\mathbb{D}$.

Either of these facts can be used to compute C_φ^* when φ is linear fractional.

Theorem 1. [Cowen 1988]
Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ *has the form*

$$\varphi(z) = \frac{az+b}{cz+d},$$

where $ad - bc \neq 0$. *Then*

$$C_\varphi^* = T_\gamma C_\sigma T_\eta^*,$$

where

$$\gamma(z) = \frac{1}{-\overline{b}z + \overline{d}},$$

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}},$$

$$\eta(z) = cz + d.$$

Proof. Consider the reproducing kernel function $K_w(z) = \frac{1}{1-\overline{w}z}$. Observe that

$$\begin{aligned}
(T_\gamma C_\sigma T_\eta^*(K_w))(z) &= \overline{\eta(w)}\gamma(z)K_w(\sigma(z)) \\
&= \left(\frac{\overline{cw}+\overline{d}}{-\overline{b}z+\overline{d}}\right)\left(\frac{1}{1-\overline{w}\left(\frac{\overline{a}z-\overline{c}}{-\overline{b}z+\overline{d}}\right)}\right) \\
&= \frac{\overline{cw}+\overline{d}}{-\overline{b}z+\overline{d}-\overline{wa}z+\overline{wc}} \\
&= \frac{1}{1-\overline{\left(\frac{aw+b}{cw+d}\right)}z} \\
&= \frac{1}{1-\overline{\varphi(w)}z} \\
&= K_{\varphi(w)}(z) = (C_\varphi^*(K_w))(z).
\end{aligned}$$

Cowen's formula can be rewritten

$$(C_\varphi^* f)(z) = \frac{(\overline{ad} - \overline{bc})z}{(\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})} f(\sigma(z)) + \frac{\overline{c} f(0)}{\overline{c} - \overline{a}z}$$

$$= \frac{z\sigma'(z)}{\sigma(z)} f(\sigma(z)) + \frac{f(0)}{1 - (\overline{a}/\overline{c})z}.$$

We would like to generalize this result so that it applies to all rational maps.

Examples of rational φ where we know C_φ^*:

(1) If $\varphi(z) = z^2$, then

$$(C_\varphi^* f)(z) = \frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}.$$

A similar formula holds for $\varphi(z) = z^m$.

(2) [Bourdon 2002] Let $\varphi(z) = \dfrac{z^2 - 6z + 9}{z^2 - 10z + 13}$.

Then

$$(C_\varphi^* f)(z) = g(z) f\left(\frac{3z - 5 - 2\sqrt{3-2z}}{9z - 13}\right)$$

$$+\frac{1}{1-z}\left[f(0) + h(z) f\left(\frac{3z - 5 + 2\sqrt{3-2z}}{9z - 13}\right)\right],$$

where

$$g(z) = \frac{2z}{\sqrt{3-2z}(4 - 3z + \sqrt{3-2z})}$$

and

$$h(z) = \frac{2z(3z - 4 - \sqrt{3-2z})}{\sqrt{3-2z}(13 - 9z)}.$$

To find a general formula, we will employ techniques developed by Cowen and Gallardo-Gutiérrez.

For any (possibly multiple-valued) analytic function g, define

$$\widetilde{g}(z) = \overline{g\left(\frac{1}{\overline{z}}\right)}.$$

Note that $\overline{g}$ and $\widetilde{g}$ agree on $\partial\mathbb{D}$.

Define $\sigma(z) = 1/\widetilde{\varphi^{-1}}(z)$.

Theorem 2. [H, Moorhouse, Robbins 2007] *Suppose that φ is a rational map that takes $\mathbb{D}$ into itself. Then*

$$(C_\varphi^* f)(z) = \sum \psi(z) f(\sigma(z)) + \frac{f(0)}{1 - \overline{\varphi(\infty)} z},$$

where

$$\sigma(z) = 1/\widetilde{\varphi^{-1}}(z),$$
$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)},$$
$$\varphi(\infty) = \lim_{|z| \to \infty} \varphi(z),$$

and the summation is taken over all branches of σ.

Our proof is divided into three cases: $|\varphi(\infty)| > 1$, $|\varphi(\infty)| < 1$, and $|\varphi(\infty)| = 1$.

Lemma 3. [Cowen and Gallardo-Gutiérrez] *Let φ be a nonconstant rational map that takes $\mathbb{D}$ into itself, with*

$$\sigma(z) = 1/\widetilde{\varphi^{-1}}(z)$$

and

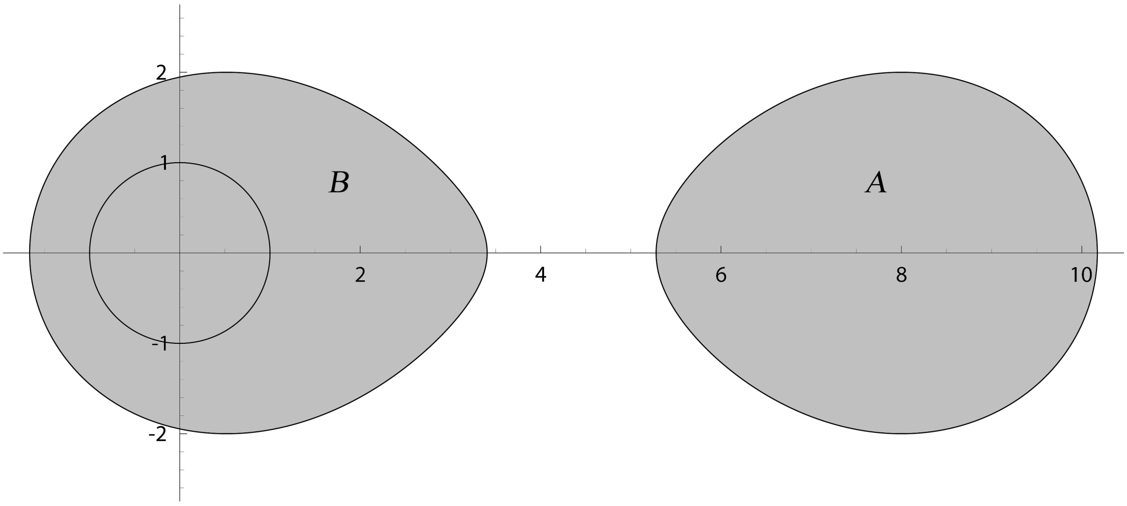
$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)}.$$

If f is a rational function with no poles on $\partial\mathbb{D}$ and g is a polynomial, then

$$\int_{\partial\varphi^{-1}(\mathbb{D})} f(\varphi(\zeta))\widetilde{g}(\zeta)\frac{d\zeta}{\zeta}$$
$$= \int_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta)g(\sigma(\zeta))}\frac{d\zeta}{\zeta},$$

where the summation is taken over all branches of σ.

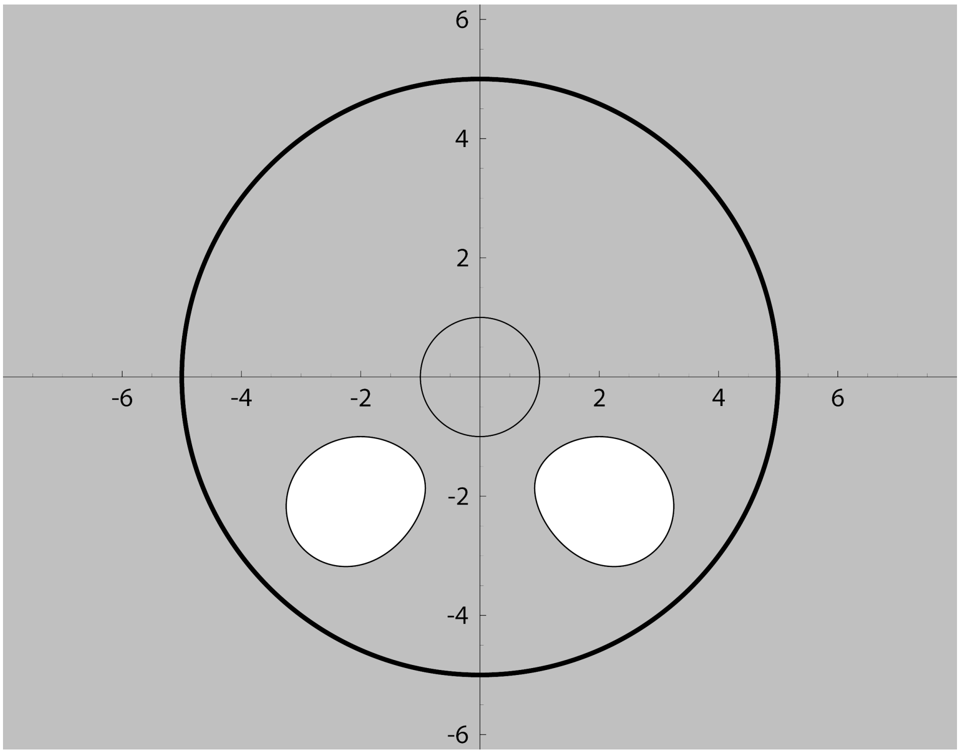
Case I: $|\varphi(\infty)| > 1$



Let f and g be polynomials. Note that

$$\begin{aligned}
\langle f, C_\varphi^*(g)\rangle &= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\varphi(\zeta))\overline{g(\zeta)}\frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\varphi(\zeta))\widetilde{g}(\zeta)\frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\int_{\partial\varphi^{-1}(\mathbb{D})} f(\varphi(\zeta))\widetilde{g}(\zeta)\frac{d\zeta}{\zeta} \\
&= \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta)g(\sigma(\zeta))}\frac{d\zeta}{\zeta}.
\end{aligned}$$

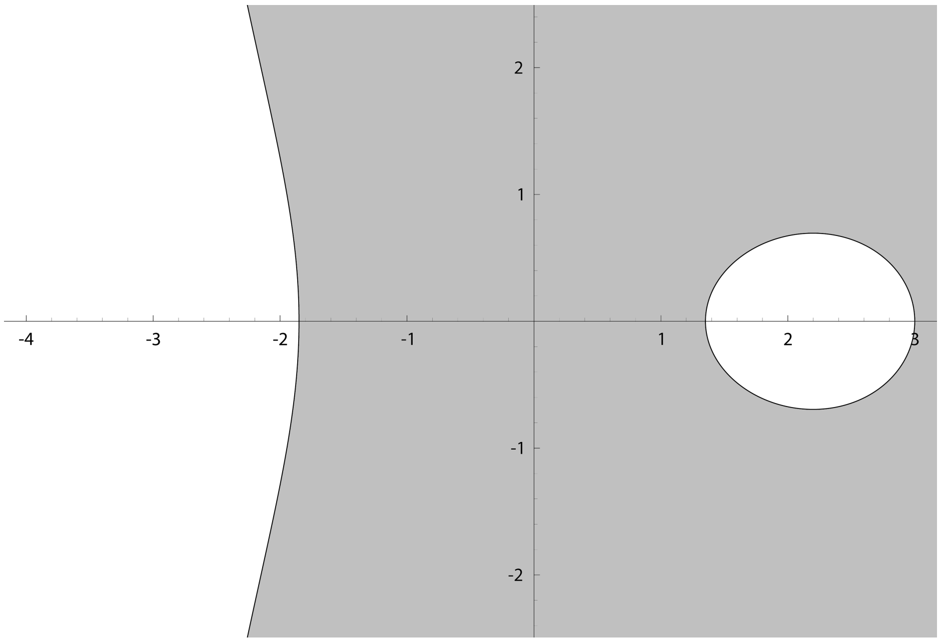
Case II: $|\varphi(\infty)| < 1$



Let f and g be polynomials. Note that

$$\langle f, C_\varphi^*(g)\rangle = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} f(\varphi(\zeta))\widetilde{g}(\zeta)\frac{d\zeta}{\zeta}$$
$$= \frac{1}{2\pi i}\int_{\mathcal{C}_R} f(\varphi(\zeta))\widetilde{g}(\zeta)\frac{d\zeta}{\zeta}$$
$$-\frac{1}{2\pi i}\int_{\partial\varphi^{-1}(\mathbb{D})} f(\varphi(\zeta))\widetilde{g}(\zeta)\frac{d\zeta}{\zeta}.$$

Case III: $|\varphi(\infty)| = 1$



Take $0 < r < 1$ and consider the map $r\varphi(z)$. Appeal to Case II.

Some open questions:

- Can one use this formula to find $\|C_\varphi\|$ when $\varphi : \mathbb{D} \to \mathbb{D}$ is a rational function?

- What can one say about C_φ^* on other spaces?