

Adjoints of Composition Operators with Rational Symbol

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Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

The *Hardy space* H^2 is the Hilbert space consisting of all analytic functions $f(z) = \sum a_n z^n$ on $\mathbb{D}$ such that

$$\|f\|_2 := \sqrt{\sum_{n=0}^{\infty} |a_n|^2} < \infty,$$

with

$$\begin{aligned} \langle f, g \rangle &:= \sum_{n=0}^{\infty} a_n \overline{b_n} \\ &= \lim_{r \uparrow 1} \int_0^{2\pi} f(re^{i\theta}) \overline{g(re^{i\theta})} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} f(e^{i\theta}) \overline{g(e^{i\theta})} \frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \overline{g(\zeta)} \frac{d\zeta}{\zeta}. \end{aligned}$$

Recall that H^2 is a *reproducing kernel Hilbert space*. That is, for any w in $\mathbb{D}$, there is a function K_w in H^2 such that

$$\langle f, K_w \rangle = f(w)$$

for all f in H^2 .

It is easy to show that

$$K_w(z) = \frac{1}{1 - \overline{w}z}.$$

Note that the span of the kernel functions is dense in H^2 .

Let φ be an analytic map from $\mathbb{D}$ into itself.

We define the *composition operator* C_φ on H^2 by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Every such operator is bounded on the Hardy space.

We would like to find a concrete representation for the adjoint C_φ^* .

A helpful observation:

$$\begin{aligned}\langle f, C_\varphi^*(K_w) \rangle &= \langle C_\varphi(f), K_w \rangle \\ &= \langle f \circ \varphi, K_w \rangle \\ &= f(\varphi(w)) \\ &= \langle f, K_{\varphi(w)} \rangle\end{aligned}$$

for all f in H^2 , so $C_\varphi^*(K_w) = K_{\varphi(w)}$.

Another useful observation:

Note that

$$\begin{aligned}(C_\varphi^* f)(w) &= \langle C_\varphi^*(f), K_w \rangle \\ &= \langle f, C_\varphi(K_w) \rangle \\ &= \langle f, K_w \circ \varphi \rangle \\ &= \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - \overline{\varphi(e^{i\theta})} w} \frac{d\theta}{2\pi}\end{aligned}$$

for all w in $\mathbb{D}$.

Either of these facts can be used to compute C_φ^* when φ is linear fractional.

Theorem 1. [Cowen 1988]

Suppose $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ has the form

$$\varphi(z) = \frac{az + b}{cz + d},$$

where $ad - bc \neq 0$. Then

$$C_\varphi^* = T_\gamma C_\sigma T_\eta^*,$$

where

$$\gamma(z) = \frac{1}{-\bar{b}z + \bar{d}},$$

$$\sigma(z) = \frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}},$$

$$\eta(z) = cz + d.$$

Proof. Consider the reproducing kernel function $K_w(z) = \frac{1}{1-\bar{w}z}$. Observe that

$$\begin{aligned}
(T_\gamma C_\sigma T_\eta^*(K_w))(z) &= \overline{\eta(w)} \gamma(z) K_w(\sigma(z)) \\
&= \left(\frac{\overline{cw} + \bar{d}}{-\bar{b}z + \bar{d}} \right) \left(\frac{1}{1 - \bar{w} \left(\frac{\bar{a}z - \bar{c}}{-\bar{b}z + \bar{d}} \right)} \right) \\
&= \frac{\overline{cw} + \bar{d}}{-\bar{b}z + \bar{d} - \overline{waz} + \overline{wc}} \\
&= \frac{1}{1 - \overline{\left(\frac{aw+b}{cw+d} \right) z}} \\
&= \frac{1}{1 - \overline{\varphi(w)} z} \\
&= K_{\varphi(w)}(z) = (C_\varphi^*(K_w))(z).
\end{aligned}$$

Cowen's formula can be rewritten

$$\begin{aligned}(C_{\varphi}^* f)(z) &= \frac{(\overline{ad} - \overline{bc})z}{(\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})} f(\sigma(z)) + \frac{\overline{c}f(0)}{\overline{c} - \overline{a}z} \\ &= \frac{z\sigma'(z)}{\sigma(z)} f(\sigma(z)) + \frac{f(0)}{1 - (\overline{a}/\overline{c})z}.\end{aligned}$$

We would like to generalize this result so that it applies to all rational maps.

Examples of rational φ where we know C_φ^* :

(1) If $\varphi(z) = z^2$, then

$$(C_\varphi^* f)(z) = \frac{f(\sqrt{z}) + f(-\sqrt{z})}{2}.$$

A similar formula holds for $\varphi(z) = z^m$.

(2) [Bourdon 2002] Let $\varphi(z) = \frac{z^2 - 6z + 9}{z^2 - 10z + 13}$.

Then

$$(C_{\varphi}^* f)(z) = g(z) f\left(\frac{3z - 5 - 2\sqrt{3 - 2z}}{9z - 13}\right) + \frac{1}{1 - z} \left[f(0) + h(z) f\left(\frac{3z - 5 + 2\sqrt{3 - 2z}}{9z - 13}\right) \right],$$

where

$$g(z) = \frac{2z}{\sqrt{3 - 2z}(4 - 3z + \sqrt{3 - 2z})}$$

and

$$h(z) = \frac{2z(3z - 4 - \sqrt{3 - 2z})}{\sqrt{3 - 2z}(13 - 9z)}.$$

To find a general formula, we will employ techniques developed by Cowen and Gallardo-Gutiérrez.

For any (possibly multiple-valued) analytic function g , define

$$\tilde{g}(z) = \overline{g\left(\frac{1}{\bar{z}}\right)}.$$

Note that $\bar{g}$ and $\tilde{g}$ agree on $\partial\mathbb{D}$.

Define $\sigma(z) = 1/\widetilde{\varphi^{-1}}(z)$.

Theorem 2. [H, Moorhouse, Robbins 2007]

Suppose that φ is a rational map that takes $\mathbb{D}$ into itself. Then

$$(C_{\varphi}^* f)(z) = \sum \psi(z) f(\sigma(z)) + \frac{f(0)}{1 - \overline{\varphi(\infty)} z},$$

where

$$\sigma(z) = 1/\widetilde{\varphi^{-1}}(z),$$

$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)},$$

$$\varphi(\infty) = \lim_{|z| \rightarrow \infty} \varphi(z),$$

and the summation is taken over all branches of σ .

Our proof is divided into three cases:

$|\varphi(\infty)| > 1$, $|\varphi(\infty)| < 1$, and $|\varphi(\infty)| = 1$.

Lemma 3. [Cowen and Gallardo-Gutiérrez]

Let φ be a nonconstant rational map that takes $\mathbb{D}$ into itself, with

$$\sigma(z) = 1/\widetilde{\varphi^{-1}}(z)$$

and

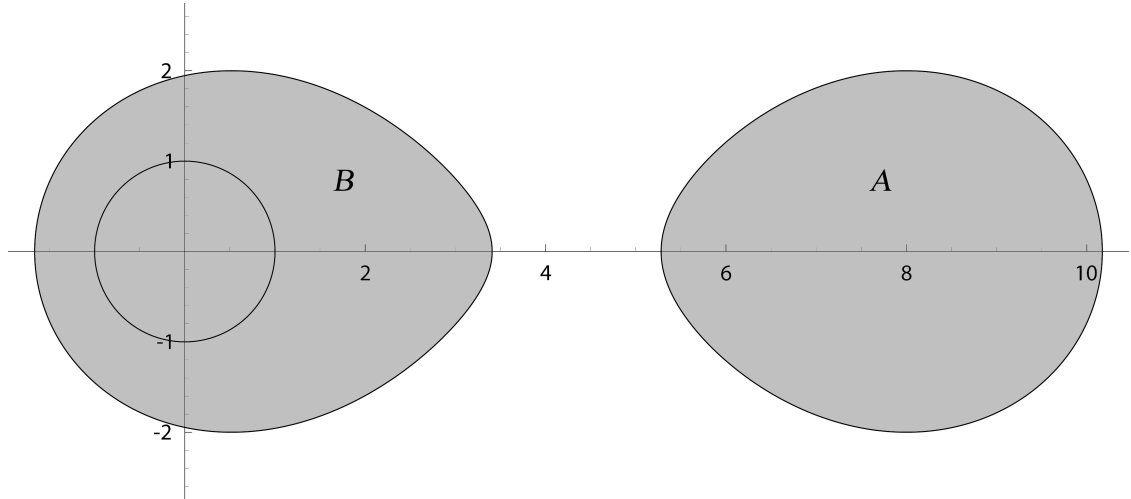
$$\psi(z) = \frac{z\sigma'(z)}{\sigma(z)}.$$

If f is a rational function with no poles on $\partial\mathbb{D}$ and g is a polynomial, then

$$\begin{aligned} \int_{\partial\varphi^{-1}(\mathbb{D})} f(\varphi(\zeta))\tilde{g}(\zeta)\frac{d\zeta}{\zeta} \\ = \int_{\partial\mathbb{D}} f(\zeta)\overline{\sum \psi(\zeta)g(\sigma(\zeta))}\frac{d\zeta}{\zeta}, \end{aligned}$$

where the summation is taken over all branches of σ .

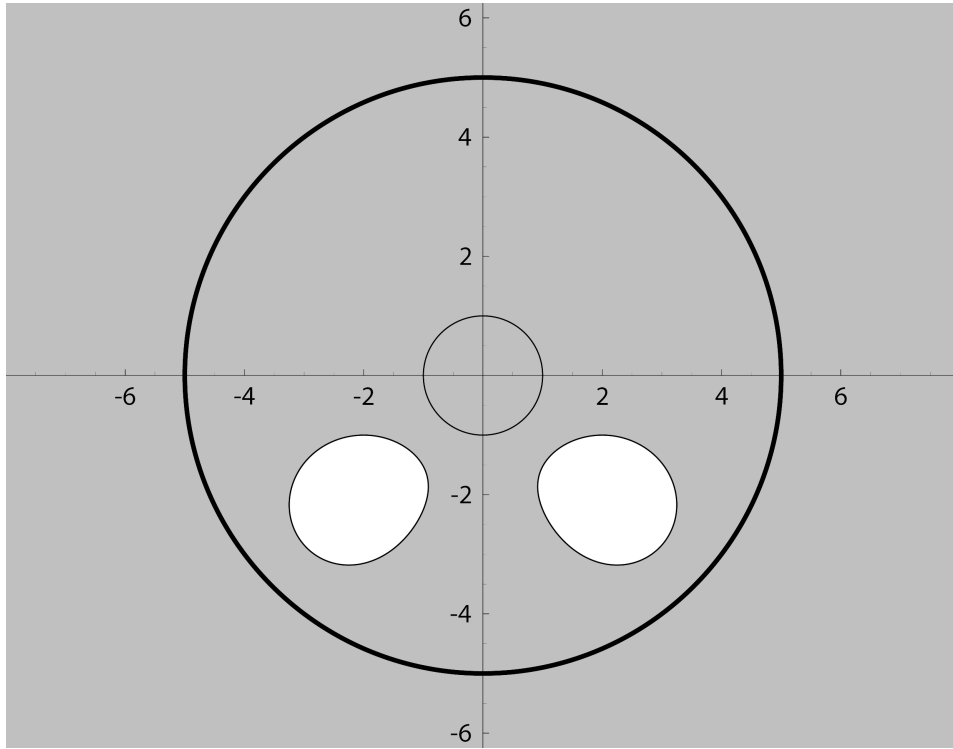
Case I: $|\varphi(\infty)| > 1$



Let f and g be polynomials. Note that

$$\begin{aligned}
 \langle f, C_{\varphi}^*(g) \rangle &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\varphi(\zeta)) \overline{g(\zeta)} \frac{d\zeta}{\zeta} \\
 &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\varphi(\zeta)) \tilde{g}(\zeta) \frac{d\zeta}{\zeta} \\
 &= \frac{1}{2\pi i} \int_{\partial \varphi^{-1}(\mathbb{D})} f(\varphi(\zeta)) \tilde{g}(\zeta) \frac{d\zeta}{\zeta} \\
 &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\zeta) \overline{\sum \psi(\zeta) g(\sigma(\zeta))} \frac{d\zeta}{\zeta}.
 \end{aligned}$$

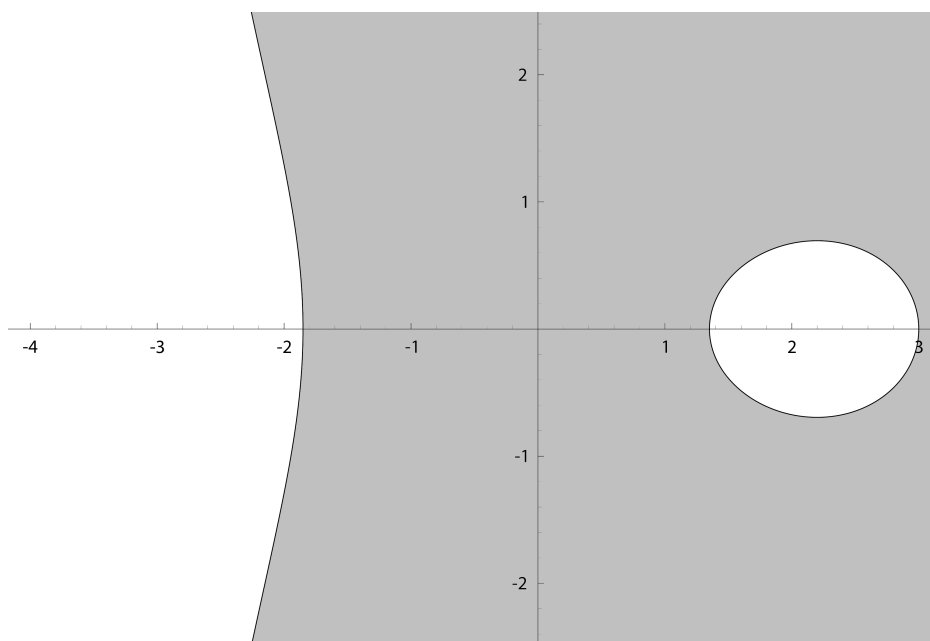
Case II: $|\varphi(\infty)| < 1$



Let f and g be polynomials. Note that

$$\begin{aligned}
 \langle f, C_{\varphi}^*(g) \rangle &= \frac{1}{2\pi i} \int_{\partial \mathbb{D}} f(\varphi(\zeta)) \tilde{g}(\zeta) \frac{d\zeta}{\zeta} \\
 &= \frac{1}{2\pi i} \int_{\mathcal{C}_R} f(\varphi(\zeta)) \tilde{g}(\zeta) \frac{d\zeta}{\zeta} \\
 &\quad - \frac{1}{2\pi i} \int_{\partial \varphi^{-1}(\mathbb{D})} f(\varphi(\zeta)) \tilde{g}(\zeta) \frac{d\zeta}{\zeta}.
 \end{aligned}$$

Case III: $|\varphi(\infty)| = 1$



Take $0 < r < 1$ and consider the map $r\varphi(z)$.
Appeal to Case II.

Some open questions:

- Can one use this formula to find $\|C_\varphi\|$ when $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is a rational function?
- What can one say about C_φ^* on other spaces?