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Zeros of Hypergeometric Functions and the Norm of a Composition Operator

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Abstract. Let φ be an analytic self-map of the unit disk; let C_φ denote the corresponding composition operator acting on the Hardy space H^2. Although the precise value of $\|C_\varphi\|$ is quite difficult to calculate, some progress has been made in the case when φ is a linear fractional map. A recent paper by Basor and Retsek demonstrates a connection between the norm of such an operator and the zeros of a particular hypergeometric series. Here we will pursue this line of inquiry further. We shall appeal to several results relating to hypergeometric series — many of which are quite old — to deduce more information about the norm of a composition operator, in particular about the spectrum of $C_\varphi^* C_\varphi$. Furthermore, we will use our knowledge of composition operators to establish an apparently new result pertaining to the zeros of hypergeometric series.

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1. Composition operators

Let $\mathbb{D}$ denote the open unit disk in the complex plane. The *Hardy space* H^2 is the Hilbert space consisting of all analytic functions $f(z) = \sum_{k=0}^\infty a_k z^k$ on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 := \sum_{k=0}^\infty |a_k|^2 < \infty.$$

For any analytic map $\varphi\colon \mathbb{D} \to \mathbb{D}$, we define the *composition operator* C_φ on H^2 by the rule

$$C_\varphi(f) = f \circ \varphi.$$

It is a consequence of Littlewood's Subordination Theorem [15, Theorem 2] that every composition operator takes H^2 boundedly into itself. Moreover, for any

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$\varphi\colon \mathbb{D} \to \mathbb{D}$, it is well known (see [5, Corollary 3.7]) that

$$\frac{1}{1-|\varphi(0)|^2} \le \|C_\varphi\|^2 \le \frac{1+|\varphi(0)|}{1-|\varphi(0)|}. \tag{1}$$

It is obvious that $\|C_\varphi\| = 1$ whenever $\varphi(0) = 0$. When $\varphi(0) \neq 0$, it is known that $\|C_\varphi\|^2$ equals the lower bound in (1) if and only if φ is a constant map ([8, Lemma 4.1], [18, Theorem 4]), and that it equals the upper bound if and only if φ is an inner function ([17, Theorem 1], [21, Theorem 5.2]). Nevertheless, there are still very few cases for which we can determine $\|C_\varphi\|$ explicitly.

The strategy for determining the norm of C_φ has often involved examining the spectrum of the operator $C_\varphi^* C_\varphi$. Recall that the spectral radius of $C_\varphi^* C_\varphi$ equals $\|C_\varphi^* C_\varphi\| = \|C_\varphi\|^2$. The following fact, which follows from an elementary Hilbert space argument (see [9, Proposition 1.2]), underscores the connection between the spectrum of $C_\varphi^* C_\varphi$ and the norm of C_φ.

Proposition 1. *Suppose that T is a bounded operator on a Hilbert space $\mathcal{H}$. Let h be an element of $\mathcal{H}$; then $\|T(h)\| = \|T\|\,\|h\|$ if and only if $(T^*T)(h) = \|T\|^2\, h$.*

In other words, C_φ attains its norm on f if and only if f is an eigenfunction for $C_\varphi^* C_\varphi$ corresponding to the eigenvalue $\|C_\varphi\|^2$. Furthermore, as long as φ is not an inner function, we know ([8, Proposition 2.3]) that such an eigenfunction cannot vanish at any point in $\mathbb{D}$.

One important consequence of Proposition 1 pertains to the relationship between the norm and essential norm of an operator. Recall that the *essential norm* $\|T\|_e$ of an operator $T\colon \mathcal{H} \to \mathcal{H}$ is defined in the following manner:

$$\|T\|_e := \inf_K \|T - K\|,$$

the infimum being taken over the set of all compact operators $K\colon \mathcal{H} \to \mathcal{H}$. Clearly $\|T\|_e \le \|T\|$ for any operator T. It follows from Proposition 1 that T is norm-attaining whenever $\|T\|_e < \|T\|$ (see [8, Proposition 2.2]). As it happens, we do have a concrete formula [20, Theorem 2.3] for the essential norm of a composition operator acting on the Hardy space. For example, if φ is a non-automorphic linear fractional map with $\varphi(1) = 1$, we know that $\|C_\varphi\|_e^2 = 1/\varphi'(1)$.

Most of the attempts to calculate the norm of a particular composition operator have, in fact, been carried out in the setting where φ is a (non-constant) linear fractional map; that is, where $\varphi\colon \mathbb{D} \to \mathbb{D}$ has the form

$$\varphi(z) = \frac{az+b}{cz+d}.$$

There have been several interesting results pertaining to such operators, most of which rely on a formula due to Carl Cowen [4, Theorem 2] for the adjoint C_φ^*. In particular, Cowen showed that $C_\varphi^* = T_\gamma C_\sigma T_\eta^*$, where

$$\sigma(z) = \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}, \qquad \gamma(z) = \frac{1}{-\overline{b}z + \overline{d}}, \qquad \eta(z) = cz + d. \tag{2}$$

In this setting, T_γ and T_η denote the analytic Toeplitz operators with symbols γ and η. Note that σ is a self-map of $\mathbb{D}$ if and only if φ has the same property (since $\sigma = \rho \circ \varphi^{-1} \circ \rho$, where $\rho(z) = 1/\overline{z}$).

As a consequence of Cowen's adjoint formula, the author [8, p. 817] observed that the operator $C_\varphi^* C_\varphi$ can be written

$$(C_\varphi^* C_\varphi f)(z) = \psi(z) f(\tau(z)) + \chi(z) f(\varphi(0)) \tag{3}$$

for all z in $\mathbb{D}$, where

$$\tau(z) = \varphi(\sigma(z)), \qquad \psi(z) = \frac{(\overline{ad} - \overline{bc})z}{(\overline{a}z - \overline{c})(-\overline{b}z + \overline{d})}, \qquad \chi(z) = \frac{\overline{c}}{-\overline{a}z + \overline{c}}.$$

This representation is valid everywhere except the point $\sigma^{-1}(0) = \overline{c}/\overline{a}$, which may or may not lie in $\mathbb{D}$. Let λ be an eigenvalue for $C_\varphi^* C_\varphi$, with corresponding eigenfunction g. Iterating equation (3), one can show [8, Proposition 5.1] that

$$\begin{aligned}\lambda^{j+1} g(0) = &\left[\prod_{m=0}^{j-1} \psi(\tau^{[m]}(\varphi(0)))\right] g(\tau^{[j]}(\varphi(0))) \\ &+ \sum_{k=0}^{j-1} \chi(\tau^{[k]}(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau^{[m]}(\varphi(0)))\right] \lambda^{j-k} g(0)\end{aligned}$$

for any integer $j \geq 0$. Here $\tau^{[k]}$ denotes the kth iterate of $\tau = \varphi \circ \sigma$; that is, $\tau^{[0]}$ is the identity map on $\mathbb{D}$ and $\tau^{[k+1]} = \tau \circ \tau^{[k]}$.

There are two recent papers that have employed this general line of reasoning. In the first, the author [8] considered the linear fractional maps φ that satisfy a particular finiteness condition: namely that $\tau^{[n]}(\varphi(0)) = 0$ for some integer $n \geq 0$. In this case, we can find a polynomial equation that allows us to determine $\|C_\varphi\|$. We state a somewhat simplified version of this result:

Theorem 2. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a linear fractional map, with $\varphi(z) \neq az$. Suppose that there is some natural number n such that $\tau^{[n]}(\varphi(0)) = 0$. Let λ be an eigenvalue for the operator $C_\varphi^* C_\varphi$ whose eigenfunctions do not vanish at the origin. Then λ is a solution to the polynomial equation*

$$\lambda^{n+1} - \sum_{k=0}^{n} \chi(\tau^{[k]}(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau^{[m]}(\varphi(0)))\right] \lambda^{n-k} = 0. \tag{4}$$

Moreover, any λ that satisfies equation (4) is an eigenvalue of $C_\varphi^* C_\varphi$.

Bourdon, Fry, Hammond, and Spofford [3] subsequently considered the more general situation, where this finiteness condition does not necessarily hold. One of their principal theorems pertains to the case where $\sup_{z\in\mathbb{D}} |\varphi(z)| = 1$; without loss of generality, we may assume that such a map fixes the point 1. With a bit of work, we may extend their result somewhat to make the following observation.

Theorem 3. *Let $\varphi\colon \mathbb{D} \to \mathbb{D}$ be a non-automorphic linear fractional map that fixes the point 1. If λ is an eigenvalue of $C_\varphi^* C_\varphi$ with $\lambda > \|C_\varphi\|_e^2$, then λ is a solution to the equation*

$$\sum_{k=0}^{\infty} \chi(\tau^{[k]}(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau^{[m]}(\varphi(0))) \right] \left(\frac{1}{\lambda} \right)^{k+1} = 1. \tag{5}$$

Conversely, any complex number $|\lambda| > \|C_\varphi\|_e^2$ that is a solution to (5) is an eigenvalue for $C_\varphi^* C_\varphi$.

When $\varphi(1) = 1$, as one would expect, Theorem 2 is simply a special case of Theorem 3. While these representations are quite helpful, it turns out that there is a particularly elegant way to rewrite equation (5). This new representation, first introduced by Basor and Retsek [2], makes it much easier to obtain information about the eigenvalues of $C_\varphi^* C_\varphi$, and hence about the norm of C_φ. In certain instances (as stated in Proposition 5), we will be able to determine the exact number of solutions to equation (5), a question that does not have an obvious answer apart from the case where $\tau^{[n]}(\varphi(0)) = 0$. In fact, when it is applicable, this result will allow us completely to determine the spectrum of the operator $C_\varphi^* C_\varphi$ (Theorem 7). In the more general setting, we will ultimately direct our attention in the opposite direction, employing our knowledge of composition operators to obtain information about the zeros of a particular class of hypergeometric series (Theorem 8).

2. Hypergeometric functions

For a, b, and c in $\mathbb{C}$, we define the *hypergeometric series*

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k,$$

where $(\,\cdot\,)_k$ denotes *Pochhammer's symbol* (also known as the *shifted factorial*):

$$(\zeta)_k := \begin{cases} 1, & k = 0, \\ \zeta(\zeta+1)\cdots(\zeta+k-1), & k = 1, 2, 3, \dots . \end{cases}$$

Note that ${}_2F_1(a, b; c; z)$ reduces to a polynomial if and only if either a or b is a non-positive integer. If c is a non-positive integer, then ${}_2F_1(a, b; c; z)$ is undefined. Except in the polynomial case, the series has radius of convergence 1, as one can see by applying the ratio test. Other basic properties of hypergeometric series are discussed in [1, Chapter 2].

Attempting to make the series representation for $\|C_\varphi\|^2$ more manageable, Basor and Retsek [2] were able to translate equation (5) into hypergeometric terms.

They observed that any non-automorphic linear fractional $\varphi\colon \mathbb{D} \to \mathbb{D}$ with $\varphi(1) = 1$ can be written

$$\varphi(z) = \frac{(\overline{\beta} - 1)z + \alpha + 1}{\overline{\alpha}z + \beta} \tag{6}$$

for certain complex numbers α and β. By translating the problem to the right half-plane, one sees that such a φ is a self-map of $\mathbb{D}$ (that fixes 1) if and only both $\overline{\alpha} + \beta$ and $\beta - \alpha - 1$ are positive real numbers. After a bit of work, Basor and Retsek obtained a much more straightforward way to write equation (5). In particular, they showed that

$$\sum_{k=0}^{\infty} \chi(\tau^{[k]}(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau^{[m]}(\varphi(0)))\right] z^{k+1} = 1 - {}_2F_1(\alpha, \beta; \delta; z/q), \tag{7}$$

where

$$\delta = \overline{\alpha} + \beta \qquad \text{and} \qquad q = \varphi'(1) = \frac{\beta - \alpha - 1}{\overline{\alpha} + \beta}.$$

In view of Theorem 3, the question of finding the norm of C_φ can be reduced to determining the zeros of the hypergeometric function ${}_2F_1(\alpha, \beta; \delta; z)$. In other words, a real number $\lambda > 1/q = \|C_\varphi\|_e^2$ is an eigenvalue for $C_\varphi^* C_\varphi$ if and only if the number $(q\lambda)^{-1} = \|C_\varphi\|_e^2 \lambda^{-1}$ in $(0, 1)$ is a zero of the hypergeometric series ${}_2F_1(\alpha, \beta; \delta; z)$. In particular, $\lambda = \|C_\varphi\|^2$, the largest eigenvalue of $C_\varphi^* C_\varphi$, corresponds to the smallest zero of ${}_2F_1(\alpha, \beta; \delta; z)$ in the interval $(0, 1)$.

Basor and Retsek used this observation to obtain a good deal of information about the norm of C_φ. In particular, for $\varphi\colon \mathbb{D} \to \mathbb{D}$ defined as in (6), they showed that equation (5) always has at least one solution, except in the case where α is a non-negative real number. They obtained this result for negative values of α by applying a limit theorem due to Gauss (see [1, Theorem 2.1.3]). For non-real values of α, they employed Pfaff's transformation (see [1, Theorem 2.2.5]) to obtain a series with real coefficients. (Basor and Retsek actually stated these results in terms of the parameter $d = \beta/\overline{\alpha}$, but their conclusions can easily be rewritten in terms of α.) The goal of this paper is to analyze more precisely the zeros of the relevant hypergeometric series. This analysis will tell us a good deal about the solutions to equation (5), and hence about the eigenvalues of $C_\varphi^* C_\varphi$.

At this point, let us pause to consider the conditions under which the hypergeometric series ${}_2F_1(\alpha, \beta; \delta; z)$ reduces to a polynomial. The conditions $\overline{\alpha} + \beta > 0$ and $\beta - \alpha - 1 > 0$, taken together, imply that $\operatorname{Re}\beta > 1/2$. Therefore we only need to consider the case where α is a non-positive integer. Observe that the condition $\alpha = 0$ corresponds to the situation where φ is an affine map. In this instance, the function ${}_2F_1(\alpha, \beta; \delta; z)$ is identically 1, so the hypergeometric series has no zeros. Hence we will only concern ourselves with the situation where α is a negative integer.

During the course of their exposition, Basor and Retsek [2, Lemma 3.2] obtained a formula for the map $\tau = \varphi \circ \sigma$ and for the iterates $\tau^{[k]}$. In terms of the parameters α and β, one can write

$$\tau^{[k]}(z) = \frac{(\beta - \alpha - 1 - k)z + k}{-kz + \beta - \alpha - 1 + k}$$

for all $k \geq 0$. Consequently

$$\tau^{[k]}(\varphi(0)) = \frac{(\beta - \alpha - 1 - k)\left(\frac{\alpha+1}{\beta}\right) + k}{-k\left(\frac{\alpha+1}{\beta}\right) + \beta - \alpha - 1 + k} = \frac{\alpha + 1 + k}{\beta + k}. \tag{8}$$

Hence we see that $\tau^{[n]}(\varphi(0)) = 0$ if and only if $\alpha = -(n+1)$. This observation hardly comes as a surprise. What we are encountering are the very cases for which the author previously discovered a polynomial representation for $\|C_\varphi\|^2$. In particular, consider the maps

$$\varphi(z) = \frac{rz - n}{-(n+1)z + r + 1}$$

for $r > n$, which are already known (see [8, Section 7]) to have the property that $\tau^{[n]}(\varphi(0)) = 0$. These maps can be rewritten in our standard form (6), with $\alpha = -(n+1)$ and $\beta = r + 1$.

We remark that, in this situation, there is another common way of writing the series ${}_2F_1(\alpha, \beta; \delta; z)$. Recall that the *Jacobi polynomial* $P_m^{(\zeta,\eta)}$ is defined in the following manner:

$$P_m^{(\zeta,\eta)}(z) := \frac{(\zeta+1)_m}{m!}\, {}_2F_1\left(-m, m + \zeta + \eta + 1; \zeta + 1; \frac{1-z}{2}\right). \tag{9}$$

(See [1, Definition 2.5.1].) Consequently, if $\alpha = -(n+1)$, a real number $\lambda > 1/q$ is an eigenvalue for $C_\varphi^* C_\varphi$ if and only if the number $(q\lambda - 2)/(q\lambda)$ in $(-1, 1)$ is a zero of the Jacobi polynomial $P_{n+1}^{(\beta-n-2,0)}(z)$. In particular, the eigenvalue $\lambda = \|C_\varphi\|^2$ corresponds to the largest zero of this polynomial within the interval $(-1, 1)$.

Before proceeding further, we mention a general result that pertains to all hypergeometric series.

Proposition 4. *Let a, b, and c be complex numbers, with c not a non-positive integer. Then ${}_2F_1(a, b; c; z)$ can have only simple zeros in $\mathbb{D}$.*

Proof. Suppose that ξ in $\mathbb{D}$ is a zero of ${}_2F_1(a, b; c; z)$ that has multiplicity 2 or greater. In other words, ${}_2F_1(a, b; c; \xi) = {}_2F_1'(a, b; c; \xi) = 0$. One of the defining properties of the hypergeometric series (see [1, Section 2.3]) is the differential equation

$$z(1-z)y'' + [c - (a + b + 1)z]y' - aby = 0. \tag{10}$$

Since ${}_2F_1(a,b;c;0) = 1$, it follows that $\xi \neq 0$; hence ${}_2F_1''(a,b;c;\xi)$ must equal 0. Differentiating equation (10), we see that ${}_2F_1'''(a,b;c;\xi) = 0$ as well. Proceeding inductively, one can show ${}_2F_1^{(k)}(a,b;c;\xi) = 0$ for any natural number k, which is impossible. ∎

The next section consists of a more detailed analysis of the zeros of certain hypergeometric series, which we use to obtain information about the eigenvalues of $C_\varphi^* C_\varphi$.

3. Real values of α

We begin by considering the case where the parameter α, as defined in (6), is real. Since $\alpha + \beta > 0$, it follows that β must be real as well, with $\beta > 1/2$. If $\alpha \geq 0$, then all of the quantities α, β, and δ are non-negative. Hence, as has already been observed (see [2, 3]), the series ${}_2F_1(\alpha,\beta;\delta;z)$ has no zeros in the interval $(0,1)$. Therefore $C_\varphi^* C_\varphi$ has no eigenvalues which are greater than $\|C_\varphi\|_e^2$, which means that $\|C_\varphi\| = \|C_\varphi\|_e$.

Suppose then that $\alpha < 0$. In this situation, Basor and Retsek [2, Proposition 4.2] observed that ${}_2F_1(\alpha,\beta;\delta;z)$ has at least one zero in the interval $(0,1)$. They divided their proof into two cases: when $\alpha \leq -1$, in which case this fact can be deduced simply by comparing the norm and the essential norm of C_φ, and $-1 < \alpha < 0$, which requires a bit more work. In the case where $-1 < \alpha < 0$, one interesting consequence of their proof is that ${}_2F_1(\alpha,\beta;\delta;z)$ has exactly one zero in the interval $(0,1)$; in other words, equation (5) has only one solution, so $C_\varphi^* C_\varphi$ has only one eigenvalue larger than $\|C_\varphi\|_e^2$. This raises an interesting question: is it possible to determine exactly how many solutions there are to equation (5)? In other words, how many eigenvalues does $C_\varphi^* C_\varphi$ have that are greater than $\|C_\varphi\|_e^2$? This question, of course, can be answered simply by examining the zeros of the series ${}_2F_1(\alpha,\beta;\delta;z)$.

The problem of determining the number of zeros of a hypergeometric series has received a good deal of attention over the years. It should come as no surprise that there are some fairly old results that provide enough information to answer the question we have just posed. The particular results we shall cite are due to Van Vleck [24], whose work is based on an earlier paper of Klein [13]. (See also a pair of papers by Hurwitz [11, 12].) In the special case of Jacobi polynomials, the pertinent facts were originally obtained by Stieltjes [23] and Hilbert [10] (see [23, Theorem 6.72] for a more modern treatment). Most of these results, in particular the ones to which we shall appeal, are stated in terms of *Klein's symbol*

$$E(u) := \begin{cases} 0, & u \leq 0, \\ \lfloor u \rfloor, & u > 0,\ u \text{ not an integer}, \\ u - 1, & u = 1, 2, 3, \ldots, \end{cases}$$

where $\lfloor\cdot\rfloor$ denotes the greatest integer function (otherwise known as the floor function). In particular, we can state the following proposition.

Proposition 5. *Let α and β be real numbers, such that $\delta=\alpha+\beta>0$ and $\beta-\alpha-1>0$. The hypergeometric series ${}_2F_1(\alpha,\beta;\delta;z)$ has exactly $E(-\alpha+1)$ zeros in the interval $(0,1)$. Consequently, if $\varphi\colon\mathbb{D}\to\mathbb{D}$ has the form*

$$\varphi(z)=\frac{(\beta-1)z+\alpha+1}{\alpha z+\beta},$$

then the operator $C_\varphi^*C_\varphi$ has exactly $E(-\alpha+1)$ eigenvalues greater than $\|C_\varphi\|_e^2$.

In other words, equation (5) has no solutions if $\alpha\geq 0$ and has exactly m solutions if $-m\leq\alpha<-m+1$. This result confirms the facts we already knew for $\alpha>-1$, but provides us with new information for $\alpha\leq -1$.

Proof of Proposition 5. Let a, b, and c be real numbers, and consider the series ${}_2F_1(a,b;c;z)$. Van Vleck treated a number of special cases, only two of which pertain to the situation we are considering. First of all, if $c>1$, the series $F(a,b;c;z)$ has the following number of zeros in the interval $(0,1)$:

$$E\left(\frac{|a-b|-|1-c|-|c-a-b|+1}{2}\right). \tag{11}$$

(See [24, p. 124].) Secondly, suppose that $\lfloor|a-b|\rfloor>\lfloor|1-c|\rfloor+\lfloor|c-a-b|\rfloor$ with $c<1$; then $F(a,b;c;z)$ has exactly

$$E\left(\frac{|1-c|+|a-b|-|c-a-b|+1}{2}\right)-E(|1-c|) \tag{12}$$

zeros in $(0,1)$. (See [24, p. 128].)

Now consider our original parameters α, β, and δ. If $\delta>1$, then expression (11) tells us that ${}_2F_1(\alpha,\beta;\delta;z)$ has exactly $E(-\alpha+1)$ zeros in $(0,1)$. Now suppose that $0<\delta<1$. Observe that $\lfloor|\alpha-\beta|\rfloor>1$, while $\lfloor|1-\delta|\rfloor$ and $\lfloor|\delta-\alpha-\beta|\rfloor$ both equal 0. Consequently expression (12) applies to this situation, giving us the same result as in the previous case: namely that ${}_2F_1(\alpha,\beta;\delta;z)$ has $E(-\alpha+1)$ zeros in the interval $(0,1)$. The case where $\delta=1$ is not explicitly addressed by Van Vleck, but one can use Hurwitz' Theorem (see [16, p. 423]) to show that the same result holds in that case. ■

Remark 1. Once we have determined the number of zeros that ${}_2F_1(\alpha,\beta;\delta;z)$ has in the interval $(0,1)$, it is natural to ask whether the series has any other zeros in $\mathbb{D}$. Similarly, since it is well known that a hypergeometric series can be analytically continued to the set $\mathbb{C}\setminus\{x\in\mathbb{R}:\ x\geq 1\}$, one might wonder how many zeros the analytic continuation of ${}_2F_1(\alpha,\beta;\delta;z)$ has throughout the complex plane. As it turns out, the function ${}_2F_1(\alpha,\beta;\delta;z)$ has no zeros outside of the interval $(0,1)$. This fact is obvious when $\alpha=-(n+1)$, since in that case ${}_2F_1(\alpha,\beta;\delta;z)$ is a polynomial of degree $n+1$. Since the function has $E(-\alpha+1)=n+1$ zeros in

$(0,1)$, it cannot have any zeros elsewhere. The case where α is not an integer requires a bit more work. Van Vleck [24] obtained several pertinent results, but there is a simpler theorem (due to Runckel [19]) that will suffice for our purposes. Suppose that $b-a\geq 0$ and $c-a-b\geq 0$. According to Runckel's theorem, the function ${}_2F_1(a,b;c;z)$ has no zeros in $\mathbb{C}\setminus\{x\in\mathbb{R}:\ x\geq 1\}$ if $a>0$; on the other hand, if $a<0$ and $c-a>0$, it has precisely

$$\lfloor -a\rfloor+\frac{1+\operatorname{sign}\{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)\}}{2}$$

zeros. (The remaining case is also dealt with, but is not of interest to us here.) Considering the parameters α, β, and δ with which we have been working, we see that in $\mathbb{C}\setminus\{x\in\mathbb{R}:\ x\geq 1\}$ the function ${}_2F_1(\alpha,\beta;\delta;z)$ has exactly $E(-\alpha+1)$ zeros.

It is worth noting that Proposition 5 has an interesting geometrical interpretation. As it turns out, the number of solutions to equation (5) depends on the position of $\varphi(0)$ relative to the point $\tau(0)=1/(\beta-\alpha)$, the center of the image disk $\varphi(\mathbb{D})$. In general, the farther $\varphi(0)$ is to the left of $\tau(0)$, the more solutions there are to equation (5).

Corollary 6. *Let α and β be real numbers, with $\alpha+\beta>0$ and $\beta-\alpha-1>0$, and consider the map*

$$\varphi(z)=\frac{(\beta-1)z+\alpha+1}{\alpha z+\beta}.$$

Equation (5) *has exactly m solutions, where m is the smallest non-negative integer such that $\tau^{[m]}(\varphi(0))\geq\tau(0)$.*

Proof. In view of (8), a simple calculation shows that

$$\tau^{[k]}(\varphi(0))-\tau(0)=\frac{\alpha+1+k}{\beta+k}-\frac{1}{\beta-\alpha}=\frac{(\alpha+k)(\beta-\alpha-1)}{(\beta+k)(\beta-\alpha)} \tag{13}$$

for any integer $k\geq 0$. Note that the quantities $\beta-\alpha-1$, $\beta+k$, and $\beta-\alpha$ are all guaranteed to be positive. Hence $\varphi(0)=\tau^{[0]}(\varphi(0))\geq\tau(0)$ if and only if $\alpha\geq 0$, in which case equation (5) has no solutions. Suppose then that $\alpha<0$; let m be the positive integer such that $-m\leq\alpha<-m+1$. It follows from (13) that m is the smallest integer for which $\tau^{[m]}(\varphi(0))\geq\tau(0)$. Moreover, Proposition 5 dictates that equation (5) has exactly m solutions. ∎

As a consequence of this corollary, we can interpret the condition $\tau^{[n]}(\varphi(0))=0$ (that is, $\alpha=-(n+1)$ for some $n\geq 0$) in a new manner. Note that this condition can be rewritten $\tau^{[n+1]}(\varphi(0))=\tau(0)$. In other words, any φ with this property can be considered an "extremal case" of the situation where $\tau^{[n+1]}(\varphi(0))\geq\tau(0)$.

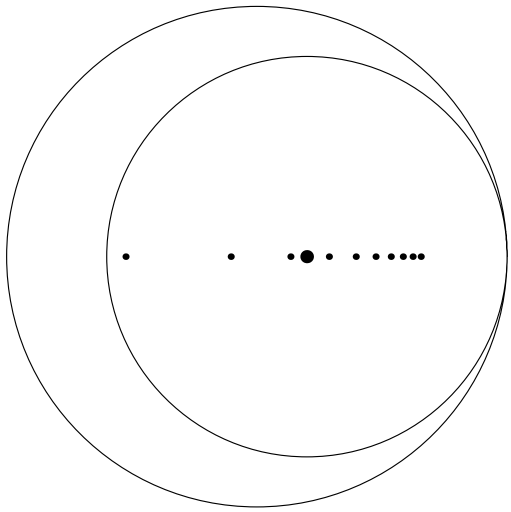

FIGURE 1. The iterates $\tau^{[k]}(\varphi(0))$.

Figure 1 serves as an illustration of Corollary 6. Consider the map

$$\varphi(z) = \frac{13z - 11}{-19z + 21}.$$

The larger disk in the figure represents $\mathbb{D}$ and the smaller disk $\varphi(\mathbb{D})$. The ten smaller points, proceeding from left to right, signify $\tau^{[k]}(\varphi(0))$ for $k = 0, 1, \ldots, 9$. The larger point denotes $\tau(0) = 1/5$, the center of $\varphi(\mathbb{D})$. Since it takes three iterations for $\tau^{[k]}(\varphi(0))$ to exceed $\tau(0)$, it follows that equation (5) has exactly three solutions. This corresponds to the fact that the value of α in this case is $-19/8$, so $E(-\alpha + 1) = 3$.

Once we have found the number of eigenvalues of $C_\varphi^* C_\varphi$, it is not difficult to determine the entire spectrum of the operator. We shall appeal to a result recently established by Kriete, MacCluer, and Moorhouse [14, Theorem 3]. Suppose that $\varphi\colon \mathbb{D} \to \mathbb{D}$ is a linear fractional map, as in (6); then there is a compact operator $K\colon H^2 \to H^2$ such that $C_\varphi^* = (1/q)C_\sigma + K$, where $\sigma\colon \mathbb{D} \to \mathbb{D}$ is the linear fractional map defined in (2) and $q = \varphi'(1)$. Hence the essential spectrum of $C_\varphi^* C_\varphi$ is identical to the essential spectrum of $(1/q)C_\sigma C_\varphi = (1/q)C_\tau$. Therefore, as noted by Kriete *et al.* [14, Corollary 3], the essential spectrum of $C_\varphi^* C_\varphi$ consists precisely of the real interval $[0, 1/q] = [0, \|C_\varphi\|_e^2]$.

Any element of the spectrum of $C_\varphi^* C_\varphi$ that is larger than $\|C_\varphi\|_e^2$, the essential spectral radius, must be an eigenvalue. In the case where α is real, we have already shown that the number of eigenvalues greater than $\|C_\varphi\|_e^2$ is precisely $E(-\alpha + 1)$. In other words, we have established the following theorem.

Theorem 7. *Let α and β be real numbers, with $\delta = \alpha+\beta > 0$ and $\beta-\alpha-1 > 0$, and consider the map*

$$\varphi(z) = \frac{(\beta-1)z+\alpha+1}{\alpha z+\beta}.$$

The spectrum of the operator $C_\varphi^* C_\varphi$ is precisely

$$\left[0, \|C_\varphi\|_e^2\right] \cup \{\lambda_k\}_{k=1}^m ,$$

where $m = E(-\alpha+1)$ and $\lambda_1, \lambda_2, \ldots, \lambda_m$ are distinct eigenvalues greater than $\|C_\varphi\|_e^2$. Furthermore, for each λ_k, the number $(q\lambda_k)^{-1} = \|C_\varphi\|_e^2 \lambda_k^{-1}$ is a zero of the hypergeometric series ${}_2F_1(\alpha, \beta; \delta; z)$.

Unfortunately, the situation for complex values of α still remains something of a mystery.

4. Complex values of α

When the parameters α and β are not real, it is difficult to make any definite statement about the number of zeros of the series ${}_2F_1(\alpha, \beta; \delta; z)$. As we have already mentioned, Basor and Retsek [2, Theorem 4.4] showed that the series must have at least one zero in the interval $(0, 1)$, except in the case where α is a positive real number. We also know that the number of zeros in $\mathbb{D}$ must be finite (see [7, pp. 99–100]). In this situation, it turns out to be easier to use composition operators to deduce information about hypergeometric series, rather than the other way around. In particular, we can obtain the following result.

Theorem 8. *Let α and β be complex numbers, with $\delta = \overline{\alpha} + \beta > 0$ and $\beta - \alpha - 1 > 0$; suppose further that α is not a non-negative real number. All of the zeros of the hypergeometric series ${}_2F_1(\alpha, \beta; \delta; z)$ within $\mathbb{D}$ must lie on the positive real axis. Moreover, the smallest such zero must be greater than or equal to*

$$\frac{(\overline{\alpha}+\beta)(|\beta|-|\alpha+1|)}{(\beta-\alpha-1)(|\beta|+|\alpha+1|)} \tag{14}$$

and less than or equal to

$$\frac{(\overline{\alpha}+\beta)(|\beta|^2-|\alpha+1|^2)}{(\beta-\alpha-1)|\beta|^2}. \tag{15}$$

Proof. Consider the map φ, as defined in (6). We know that φ must take $\mathbb{D}$ into itself, with $\varphi(1) = 1$ and

$$q = \varphi'(1) = \frac{\beta-\alpha-1}{\overline{\alpha}+\beta} > 0.$$

Suppose that a point ξ in $\mathbb{D}$ is a zero of ${}_2F_1(\alpha, \beta; \delta; z)$. As a consequence of Theorem 3 and equation (7), we see that $(q\xi)^{-1}$ must be an eigenvalue of $C_\varphi^* C_\varphi$; hence ξ must be a positive real number. We know that the smallest zero of

${}_2F_1(\alpha,\beta;\delta;z)$ corresponds to the largest eigenvalue of $C_\varphi^* C_\varphi$, that is, to $\|C_\varphi\|^2$. Equation (1) provides us with an upper and lower bound for the $\|C_\varphi\|^2$. Recalling that $\varphi(0) = (\alpha+1)/\beta$, we obtain the stated bounds for the smallest zero of ${}_2F_1(\alpha,\beta;\delta;z)$. ■

If $\operatorname{Re}\alpha > -1/2$, it is possible for the quantity in expression (15) to be greater than 1; hence the upper bound does not always provide us with useful information. Expression (14), however, is always strictly greater than 0. It appears that the result of Theorem 8 has not been known previously, and that it would be difficult to prove using the conventional techniques associated with hypergeometric functions.

Combining all the information we have obtained so far, we can state a version of this result that pertains to a particular class of Jacobi polynomials.

Corollary 9. *Let m be a positive integer and let ζ be a real number greater than -1. The Jacobi polynomial $P_m^{(\zeta,0)}(z)$ has precisely m zeros in the interval $(-1,1)$, the largest of which is greater than or equal to*

$$\frac{(\zeta+m+1)^2 - 2(\zeta+1)(\zeta+2)}{(\zeta+m+1)^2}$$

and less than or equal to

$$\frac{(\zeta+2m)^2 - 2(\zeta+1)(\zeta+2)}{(\zeta+2m)^2}.$$

Proof. Let $\alpha = -m$ and $\beta = \zeta + m + 1$. It follows from our assumptions that $\alpha + \beta = \zeta + 1 > 0$ and $\beta - \alpha - 1 = \zeta + 2m > 0$. Hence Proposition 5 dictates that ${}_2F_1(\alpha,\beta;\delta,z)$ has m zeros in $(0,1)$. Similarly, Theorem 8 provides us with an upper and lower bound for the smallest of these zeros. Recalling the correspondence between Jacobi polynomials and hypergeometric series, as shown in (9), we obtain the stated results. ■

Remark 2. The most obvious question that remains to be answered is how many solutions equation (5) has when α and β are not real. In this setting, it would also be desirable to provide some sort of geometric interpretation, akin to Corollary 6, for the number of solutions. In addition, there are several other lines of inquiry one might pursue. For example, one could consider linear fractional $\varphi\colon \mathbb{D} \to \mathbb{D}$ with $\sup_{z\in\mathbb{D}} |\varphi(z)| < 1$. For some of these maps, such as $\varphi(z) = 1/(3-z)$, it has been shown [3, Theorem 4.3] that the norm of C_φ is still given by equation (5). On the other hand, there are many maps, for instance $\varphi(z) = (4z+4)/(z+12)$, for which equation (5) provides no information. In this situation, there is no obvious way to rewrite (5) in terms of hypergeometric series. It would be desirable to find an alternate representation that would allow us to determine when (5) has a solution, and to obtain further information about the spectrum of $C_\varphi^* C_\varphi$.

Certain norm calculations have recently been carried out in the setting where φ is a rational function (see [6]). It would also be interesting to obtain a general representation for the eigenvalues of $C_\varphi^* C_\varphi$ that holds in such cases as well.

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